An Embedded Markov Chain Approach to Stock Rationing with Batch Orders

Mehmet Murat Fadıloğlu[a], Önder Bulut[b]

[a] *Department of Industrial Systems Engineering, Izmir University of Economics, Izmir, Turkey*

murat.fadiloglu@ieu.edu.tr

[b] *Department of Industrial Engineering, Yaşar University, Izmir, Turkey*

onder.bulut@yasar.edu.tr

Abstract

Rationing is an inventory policy that allows prioritization of demand classes. It enables the inventory system to provide higher service levels for critical demand classes. In this paper, we propose a new method for the analysis of the backordering inventory systems under rationing with batch orders. We show that if such an inventory system is sampled at multiples of supply lead-time, the state of the system evolves according to a Markov chain. We provide a recursive procedure to generate the transition probabilities of this embedded chain. Although the embedded Markov chain has an infinite state space, it is possible to obtain the steady-state probabilities of interest with desired accuracy by considering a truncated version of the chain. The obtained probabilities are also the steady-state probabilities of the original continuous-time system and permit the computation of any long-run performance measure of interest.

Keywords: Inventory; rationing; embedded Markov chains; infinite state Markov chains.

1. Introduction

Companies often wish to provide different service levels to different customers in order to achieve higher operational efficiency. The underlying motivation usually is the difference in service level requirements or shortage costs for customer classes. When a company does not differentiate the service it provides to different customers, this results in higher inventory carrying costs. The practice of rationing inventory among customer classes is a well-known tool to increase operational efficiency.

Rationing mechanism stops serving lower priority classes when the inventory on hand drops below some critical level. The remaining stock is reserved for higher priority classes. If there are more than two demand classes, other critical levels are used to differentiate them. The same critical levels are also often used for clearing the backorders. That is, backorders for a certain demand class cannot be cleared until the on hand inventory reaches to the critical level for that class. Below this level, the incoming replenishment batches are either used to clear the backorders of more critical classes (for which the associated critical levels are lower) or to increase the on hand inventory. This clearing mechanism is called the priority clearing in the literature.

The above described rationing policy usually uses static (constant) critical levels. The bulk of the literature in the area of rationing is dedicated to the analysis of this static rationing policy. However, it is a well-known fact that the static rationing policy is not optimal. Under the optimal policy, the critical levels would change dynamically according to the number and the times-to-arrive of all outstanding replenishment orders. Specifically, the rationing levels would decrease as a replenishment order is about to arrive, since the likelihood of having a stockout in the remaining time before the replenishment is relatively small. Yet such a dynamic policy has hardly been touched in the literature, since the structure of the optimal policy is still unknown. Even if a policy structure is assumed, its analysis would be prohibitively difficult. Due to this inherent difficulty, Fadıloğlu and Bulut (2010a) is the only study that considers the analysis of the much simpler (compared to the dynamic policies) static rationing policy under priority clearing. For a more detailed discussion on the structure of the optimal rationing policies in different inventory and production settings, we direct the reader to the recent work of Bulut and Fadıloğlu (2011).

For a two-class-system, Fadıloğlu and Bulut (2010b) provide a numerical study which shows that if the demand rates of the customer classes are close to each other, the benefit of the static rationing under priority clearing over the FCFS policy increases. They also propose two lower bounds on the performance of the unknown optimal policy, and demonstrate that when the demand is mostly from the higher priority class, it is not possible to obtain any extra benefit over static rationing even with the optimal policy. Deshpande et al. (2003) also compares the performance of the static policy (under a different clearing mechanism they construct to make the analysis possible) with another lower bound and conclude that the performance gap is small for cost and demand parameters typical of military service parts.

All these observations indicate that the static rationing, which is a relatively simpler policy – in terms of application and analysis–, is a meaningful policy that is not easy to improve upon. This is also in agreement with the interest in the policy both in practice and the literature. Although there are settings in which there may be benefit to consider more sophisticated dynamic policies, the static policy is likely to maintain its importance in customer differentiation.

Fadıloğlu and Bulut (2010a) present a new method for the analysis of continuous-review inventory systems with backordering, lot-per-lot ordering policy, static rationing, and priority clearing. In this paper, we extend the study of Fadıloğlu and Bulut (2010a) and provide an analysis under batch orders, i.e., under (Q,R) ordering policy. Instead of ordering one replenishment unit every time a demand occurs, the inventory system waits until Q units of demand occur and then dispatches a replenishment order of Q units. For this setting that the rationing mechanism does not have any effect on the ordering dynamics. But unfortunately, the converse statement does not hold, i.e., the ordering policy affects the rationing mechanism and thereby the backlog in the inventory system.

A two-dimensional state-space is enough to capture the system dynamics under the $(S-1,S)$ policy. However, in the case of the (Q,R) policy, the analysis requires a three-dimensional state-space representation that incorporates a new counter variable. Furthermore, we have to modify the definition of another state variable by using a new concept that we name "replenishment opportunity". The underlying model is challenging and generalizing the analysis to batch orders was only possible after the introduction of these innovations.

Our method is based on the observation that the state of the inventory system sampled at multiples of the supply lead-time evolves according to a Markov chain. Our analysis yields an approximation for the steady-state distribution for the inventory system, which can be used to obtain any long-run performance measure. Although there are other approximate results in the literature, our approach is the only one that takes into account the priority clearing mechanism. Deshpande et al. (2003) address the importance of priority clearing on page 684 of their study: "The optimal scheme is to always clear higher-priority customers first. However, this "priority-clearing" scheme is intractable because it does not allow closed-form expressions for the stock-out levels, and average number of demand in backlog, for each demand class. To overcome this problem we introduce a tractable "threshold clearing" scheme to approximate the systems dynamics." A steady-state analysis for inventory systems under batch ordering policy and static rationing with priority clearing was not available up to this point. This is the main contribution of this study.

The research in the area of rationing was initiated in the 1960's. Veinott (1965) is the first to analyze multiple demand classes in a periodic-review setting with backordering. He introduces the use of critical levels in providing different service levels to different demand classes. A recent work in a periodic-review setting by Mollering and Thonemann (2010) also considers the priority clearing dynamics. But, the dynamics are quite different in the periodic-review setting.

Nahmias and Demmy (1981) are the first to consider a continuous-review environment. They assume two demand classes with Poisson arrivals, constant lead-time and full backordering for performance evaluation purposes. They derive approximate service level expressions under a (Q, R, K) policy, which is a (Q, R) policy with a fixed rationing level. Their approximation is based on the at-most-one-order-outstanding assumption. Nahmias and Demmy (1981) apply the same analysis to a periodic setting as well in the same paper.

Dekker et al. (1998) focus on the same setting with Nahmias and Demmy (1981) with the exception of considering a lot-per-lot ordering policy instead of (Q, R) policy. They also derive approximate service level expressions. Kocaga and Sen (2007) extend the approximation of Dekker et al. (1998) to accommodate a demand lead-time. Deshpande et al. (2003) consider the same setting with Nahmias and Demmy (1981). They derive approximate service level expressions that are exact under a threshold clearing mechanism they construct to make the analysis possible. It is interesting that their analysis yields the same results with Dekker et al. (1998) when Q =1. They also provide an algorithm to find the optimal policy parameters under a given cost structure. Teunter and Haneveld (2008) and Fadıloğlu and Bulut (2010b) investigate dynamic rationing strategies for the same setting.

Arslan et al. (2007) also consider the same setting. But, instead of working directly on the original single-location system, they construct an equivalent multi-stage serial inventory system. The multi-stage serial system clears backorders at each stage in the order of occurrence. Hence, the proposed equivalence does not assume priority clearing for the original system, but some other clearing mechanism as in the case of Deshpande et al (2003). The strength of their approach is that it is directly applicable to more than two demand classes.

Dekker et al. (2002) consider a lot-per-lot continuous-review setting in a lost sales environment. The clearing mechanism issue is not relevant for the lost sales case. They provide exact expressions for service levels under general stochastic lead-time and multiple demand classes. Their results are adapted from the analysis of $M/G/\infty$ queue under state-dependent arrival rates.

A parallel line of research looks at the same problem in continuous time under a capacitated replenishment channel. Ha (1997a) considers a make-to-stock production facility, multiple demand classes and a lost sales environment. The production is performed by a single exponential server. Using uniformization technique, he shows that a critical level policy is optimal. Ha (1997b) extends his previous findings to backordering environments, while Ha (2000) shows the optimality of a critical level policy based on the work storage level he defines under Erlang production times in a lost sales environment. Vericourt et al (2002) extends the findings of Ha (1997b) to multiple demand classes. Gayon et al. (2009) considers Erlangian production times in a backordering environment. Bulut and Fadıloğlu (2011) introduce parallel production channels.

In Section 2, we provide the method to obtain an embedded Markov chain for a system under (Q, R) ordering policy and static rationing with priority clearing. The method is based on a recursive procedure, which generates the one-step transition probabilities of the embedded Markov chain corresponding to the system under consideration. Section 3 is devoted to steady-state analysis of the chain. In this section, we show that we can get steady-state probabilities of interest with desired accuracy by considering transition probabilities corresponding to a subset of the state space. The application of this technique is mandatory for our problem since the state space for the embedded Markov chain is infinite. In Section 4, we demonstrate that the technique converges to acceptable accuracy levels fairly quickly and compare our results with the previously available techniques. The paper concludes in Section 5 with a discussion of possible extensions.

2. The Embedded Markov Chain

We consider an inventory system that experiences demands from two customer classes according to two independent Poisson processes with rates λ_1 and λ_2. This means that the total demand also follows a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. For this system, inventory is replenished according to (Q, R) policy. That is, the inventory position of the system hits R+Q every time a replenishment order is placed and then it goes down one by one with every demand occurrence until it hits R+1. The next demand occurrence at the level R+1 triggers a replenishment order and brings the inventory position again to R+Q. Replenishment orders arrive after a deterministic lead-time $L > 0$.

If there is inventory on-hand, arriving Class 1 demands are instantaneously satisfied. They are only backordered when on-hand inventory depleted, i.e., the rationing level for class 1 is zero. Arriving class 2 demands are instantaneously satisfied if the on-hand inventory level is above the critical rationing level of class 2, which is denoted as *K*. Otherwise, class 2 demands are backordered. Backorders are cleared according to the priority clearing mechanism. That is, when a replenishment order arrives, it is first used to clear class 1 backorders then inventory level is increased up to *K* and at this level, the remaining replenishment quantity (if any remains) is used to clear class 2 backorders. The inventory level can be increased above *K* only after clearing all class 2 backorders.

To keep track of the system's behavior in time, we propose a three-dimensional state-space. We define the state of the system at time *t* as $\left(X(t), C(t-L), B(t)\right)$. We use the two of the state variables $X(t)$ and $B(t)$, which were also used in the lot-per-lot model of Fadıloğlu and Bulut (2010a). However, we need to modify the definition of the state variable $X(t)$ for the (Q, R) model. Here, $X(t)$ is the number of demand arrivals of both types in $(t-L, t]$. This is not the number of outstanding replenishment orders at time *t*, since every demand arrival does not trigger a replenishment order unlike in the lot-per-lot case. $B(t)$ is still the number of class 2 backorders at time *t*. We also define $C(t)$ as the count of demand arrivals since the last replenishment order. Hence, $C(t-L)$ is the demand count a lead-time before the current point in time.

Obviously $0 \le C(t) \le Q-1$, since at the Q^{th} demand arrival after last replenishment order, a new replenishment order is given. Furthermore, we can relate the counts of the system at multiples of the lead-time with

$$C(t) = \left(C(t-L) + D(t-L, t]\right) \bmod Q = \left(C(t-L) + X(t)\right) \bmod Q . \tag{1}$$

Equation (1) is based on the fact that each demand arrival at which no replenishment order is given increases the counter by one and demand arrivals causing replenishment orders zero the counter. One can express the inventory position of the system at time *t* as a function of $C(t)$ as

$$IP(t) = R + Q - C(t), \tag{2}$$

since the inventory position hits R+Q at the times replenishments are ordered and then goes down by one each time a demand occurs. The demand $(t-L, t]$ decreases the inventory level from the inventory position at time t-L or causes a class 2 backorder, i.e.,

$$I(t) = IP(t-L) - X(t) + B(t) \ . \tag{3}$$

Using (2) to express the inventory position at t-L, (3) transforms into

$$I(t) = R + Q - C(t-L) - X(t) + B(t) . \tag{4}$$

This means that we are able to express the inventory level at any time in terms of the state variables.

There is no backorder if the inventory level is above *K*, since all backorders need to be cleared at the support level, i.e.,

$$B(t) = 0 \ \text{if} \ X(t) + C(t-L) < R + Q - K \,. \tag{5}$$

If the inventory level is at *K* or lower, only class 2 demands that arrive after the inventory level hits *K* would be backordered. Even if all the demand arrivals belong to class 2, the maximum number of backorders would be $X(t) + C(t-L) - (R+Q-K)$. Thus,

$$0 \le B(t) \le X(t) + C(t-L) - (R+Q-K) \ \ \text{if} \ X(t) + C(t-L) \ge R + Q - K. \tag{6}$$

Conditions (5) and (6) specify the feasible states and thereby the state space of the embedded Markov chain, while Equation (4) specifies the inventory level corresponding to each state of the state space.

Given the state of the system at *t*, it is possible to derive the probability that the system reaches a certain state at time *t*+*L*. Once these probabilities are obtained for all feasible states at time *t*, and at time *t*+*L*, an embedded Markov chain, which models the evolution of the system at multiples of the lead-time, is obtained. The underlying continuous-time process describing the evolution of the system is regenerative, since the process regenerates itself every time the inventory level of the system is at level *R*+*Q*, i.e., when $\left(X(t), C(t-L), B(t)\right) = (0,0,0)$. Thus, the process is ergodic and the limiting distribution of the process exists (see Stidham (1974)). That is for any ε value there exist a t_ε such that

$$\left| P\left\{X(t) = x, C(t-L) = c, B(t) = b\right\} - P\left\{X = x, C = c, B = b\right\} \right| \le \varepsilon \ \text{for all} \ t \ge t_\varepsilon \,.$$

Moreover, the limiting distribution of the embedded Markov chain is the same with the one of the underling continuous-time process since

$$\left| P\left\{X(nL) = x, C((n-1)L) = c, B(nL) = b\right\} - P\left\{X = x, C = c, B = b\right\} \right| \le \varepsilon \quad \text{for all} \ \ n \ge \left\lceil \frac{t_\varepsilon}{L} \right\rceil .$$

The state variable $X(t)$, which is the number of demand arrivals in $(t-L, t]$, is independent of the system dynamics and is a Poisson random variable with mean λL. Thus, when sampled at multiples of lead-time, $X(t)$ evolves according to an embedded Markov chain whose one-step transition probabilities are independent of the origin state and the rationing policy, i.e.

$$P\left\{X(t+L)=x_L \mid X(t)=x_0\right\}=P\left\{D_{(t,t+L]}=x_L\right\}=e^{-\lambda L}\frac{(\lambda L)^{x_L}}{x_L!}\quad for\ x_L=0,1,2,... \tag{7}$$

Moreover, the count of the system at multiples of the lead-time is determined by the number of demand arrivals during that lead-time as stated in (1), thereby

$$\text{for } x_L=0,1,2,...,\quad P\left\{X(t+L)=x_L, C(t)=c_L \mid X(t)=x_0, C(t-L)=c_0\right\}= \begin{cases} e^{-\lambda L}\dfrac{(\lambda L)^{x_L}}{x_L!} & \text{for } c_L=(c_0+x_0)\bmod Q \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

This result actually describes the evolution of the inventory systems under (Q, R) policy at multiples of the lead-time and is totally independent of the rationing policy.

The one step transition probabilities for the system can be expressed as $P\left\{X(t+L)=x_L, C(t)=c_L, B(t+L)=b_L \mid X(t)=x_0, C(t-L)=c_0, B(t)=b_0\right\}$. The probabilities that relate to event of reaching an inventory level above K can be obtained directly from (8) as

$$P\left\{X(t+L)=x_L, C(t)=c_L, B(t+L)=0 \mid X(t)=x_0, C(t-L)=c_0, B(t)=b_0\right\}= \begin{cases} e^{-\lambda L}\dfrac{(\lambda L)^{x_L}}{x_L!} & \text{for } c_L=(c_0+x_0)\bmod Q \\ 0 & \text{otherwise} \end{cases}, \tag{9}$$

$$\text{for } 0\le x_L+c_L\le R+Q-K,\ \text{for all feasible}\,(x_0,c_0,b_0),$$

since there will be no backorders when the inventory level is above the support level.

One needs considerably more effort in order to obtain other one-step transition probabilities. Since we know the distribution of $X(t+L)$ and the fact that the distribution is independent of the state at time t, i.e., $\left(X(t), C(t-L), B(t)\right)$, we can use this to write

$$P\left\{X(t+L)=x_L, C(t)=c_L, B(t+L)=b_L \mid X(t)=x_0, C(t-L)=c_0, B(t)=b_0\right\}= P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{matrix} X(t+L)=x_L, X(t)=x_0, \\ C(t-L)=c_0, B(t)=b_0 \end{matrix}\right\} e^{-\lambda L}\frac{(\lambda L)^{x_L}}{x_L!}. \tag{10}$$

Thus, we need to compute

$$P\left\{C(t)=c_L, B(t+L)=b_L \mid X(t+L)=x_L, X(t)=x_0, C(t-L)=c_0, B(t)=b_0\right\} \tag{11}$$

for all feasible state triplets $\left(x_0, c_0, b_0\right)$ and $\left(x_L, c_L, b_L\right)$.

In order to do this computation, we would need a modified version of Proposition 2 in Fadıloğlu and Bulut (2010a). This proposition for the lot-per-lot model states that given the number of demand arrivals in $(t-L,t]$ and in $(t,t+L]$, the replenishment times and the demand arrival times in $(t,t+L]$ are iid random variables with uniform distribution over $(t,t+L]$. However, for the (*R*, *Q*) case, the proposition does not hold as is, since each demand arrival in $(t-L,t]$ does not trigger a replenishment order. To overcome this difficulty, instead of keeping track of the replenishments, we keep track of –what we call– replenishment opportunities. Each time a demand arrives; it triggers a replenishment opportunity that will realize exactly after one lead-time. In the (*S*-1, *S*) case, each replenishment opportunity corresponds to an actual replenishment arrival. In contrast, in the $\left(Q,R\right)$ case, out of every *Q* consecutive replenishment opportunity, only one corresponds to an actual replenishment arrival –of size *Q*–. Thus, when we look at the system in terms of replenishment opportunities instead of replenishment arrivals, Proposition 2 of Fadıloğlu and Bulut (2010a) still holds: Given the number of demand arrival in $(t-L,t]$ and in $(t,t+L]$, the replenishment opportunity times and the demand arrival times are independently and uniformly distributed over $(t,t+L]$. However, one needs to keep track of which opportunity triggers an actual replenishment.

When the number or order arrivals and replenishment opportunities during $(t,t+L]$ is known, it is their relative order that determines the number of backorders at time $t+L$. We propose a recursive procedure to compute the probabilities of interest expressed in (11). This procedure is based on the following expression:

$$P\left\{C\left(t\right)=c_L, B\left(t+L\right)=b_L \mid X(t+L)=x_L, C\left(t'-L\right)=c, B\left(t'\right)=b, Y(t')=y, Z(t')=z\right\} \quad \text{for } t \le t' \le t+L \tag{12}$$

where *Y(t')* is the number of demand arrivals in $(t',t+L]$, $Z(t')$ is the number of replenishment opportunities in $(t',t+L]$, and $C(t'-L)$ is the count of the process a lead-time ago. We keep track of $C(t'-L)$ since it is the indicator that distinguishes the actual replenishment times from empty replenishment opportunities. $(t',t+L]$. Here we have to use *t'* instead of *t*, since we need probabilities on what happens between any point in time -- within the lead-time period -- and the end of the period. We can express (11) in terms of (12) as

$$P\left\{C(t)=c_L, B(t+L)=b_L \mid X(t+L)=x_L, X(t)=x_0, C(t-L)=c_0, B(t)=b_0\right\} = P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(t-L)=c_0, \\ B(t)=b_0, Y(t)=x_L, Z(t)=x_0 \end{array}\right\} \quad . \tag{13}$$

In order to devise a method to compute the probabilities of (12), we condition them on the first occurrence. This can either be a demand arrival or a replenishment opportunity. We define $f(u \mid b, y, z, t')$ as the density function for the first occurrence time after t', given that there are b units of backorder at time t', and the number of demand arrivals and replenishment opportunities to occur in $(t', t+L]$ are y and z, respectively. We also define $P\left\{A(u)=a \mid b, y, z, t'\right\}$ where $a \in \{d, r\}$ as the probability that the first occurrence, which takes place at time u, is a replenishment opportunity or a demand arrival under the same conditions.

THEOREM 1. Let $p_1 = \lambda_1 / (\lambda_1 + \lambda_2)$, $p_2 = 1 - p_1$, $x = x_L + z - y$, $y \geq 0$, $z \geq 0$ and $b \in \left[0, \left((x+c)-(R+Q-K)\right)^+\right]$. The following recursive equation holds for $\max(y,z) \neq 0$:

$$P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(t'-L)=c, \\ B(t')=b, Y(t')=y, Z(t')=z \end{array}\right\} =$$

$$\int_{u=t'}^{t+L} \left[P\left\{A(u)=r \mid b,y,z,t'\right\} P\left\{ \begin{array}{c} C(t)=c_L, \\ B(t+L)=b_L \end{array} \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(u-L)=(c+1) \bmod Q, \\ B(u)=\left(b-\left(Q-\left((x+c-b)-(R+Q-K)\right)\right)^+ \mathbf{1}\{c=Q-1\}\right)^+, \\ Y(u)=y, Z(u)=(z-1)^+ \end{array} \right\} \right. \tag{14}$$

$$+P\left\{A(u)=d \mid b,y,z,t'\right\}\left(p_1 P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(u-L)=c, \\ B(u)=b, Y(u)=(y-1)^+, Z(u)=z \end{array}\right\}\right.$$

$$\left.\left. + p_2 P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(u-L)=c, \\ B(u)=b+\mathbf{1}\{x+c-b \geq R+Q-K\}, \\ Y(u)=(y-1)^+, Z(u)=z \end{array}\right\}\right)\right] f(u \mid b,y,z,t')\,du$$

Moreover, the following boundary condition holds:

$$P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{array}{c} X(t+L)=x_L, C(t'-L)=c, \\ B(t')=b, Y(t')=0, Z(t')=0 \end{array}\right\} = \mathbf{1}\{b_L=b\}.\mathbf{1}\{c_L=c\} \tag{15}$$

PROOF: In order to determine the system's behavior to any event (demand arrival or replenishment arrival), we need to know the inventory level of the system or, equivalently, the number of demand arrivals in $(t'-L,t']$, $X(t')$. Given $X(t+L)=x_L, Y(t')=y, Z(t')=z$, $X(t')$ is determined by

$$X(t')=X(t+L)+Z(t')-Y(t')=x_L+z-y \tag{16}$$

and we denote the value of $X(t')$ as *x*.

The theorem relates conditional probabilities that specify the state of the system at time *t'*. Thus, it is applicable only for the values of the state variables, which correspond to feasible states. Since *x* is the number of replenishment orders at *t'*, (5) and (6) yield the condition $b\in\left[0,\left((x+c)-(R+Q-K)\right)^{+}\right]$.

The theorem is based on the law of total probability. We condition the probability of interest on the time and the type of the first occurrence in $(t',t+L]$. That is, the probability that the first arrival in $(t',t+L]$ is at time *u* and is a replenishment opportunity is $P\{A(u)=r\,|\,b,y,z,t'\}.f\left(u\,|\,b,y,z,t'\right)du$, and the probability that the first arrival in $(t',t+L]$ is at time *u* and is a demand arrival is $P\{A(u)=d\,|\,b,y,z,t'\}.f\left(u\,|\,b,y,z,t'\right)du$.

The effect of this first occurrence depends on the inventory level at the occurrence time as a result of rationing policy. The inventory level can be expressed as $R+Q-x-c+b$ using (4). Thus, $x+c-b$ determines the state of the system at the occurrence time.

When a replenishment opportunity occurs first, it may either correspond to an empty opportunity or an actual replenishment arrival depending on the replenishment indicator, $C(t'-L)$. If the indicator is less than $Q-1$, the opportunity is empty and we only increase the replenishment indicator since the current replenishment opportunity is consumed. If the indicator is equal to $Q-1$, the opportunity corresponds to an actual replenishment arrival of size Q. In this case, if there is any class 2 backorder to be cleared –in case of which the inventory level is at or below the rationing support level–, number of units left in the order after bringing the inventory level to the clearing position is $\left(Q-\left((x+c-b)-(R+Q-K)\right)\right)^{+}$. This quantity is subtracted from the current class-2-backorder level. If the result is negative or zero, then there is no class 2 backorder left, i.e., $B(u)=0$.

When a demand arrival of class 1 occurs first, the number of demand arrivals to occur is decreased by one. Finally, when a demand arrival of class 2 occurs first, the number of demands to arrive is again decreased by one, and the number of class 2 backorders is increased if the inventory level is at or below the support level, i.e., $x+c-b \geq R+Q-K$.

The theorem also includes Equation (15), which determines the probabilities corresponding to the case at which neither a demand nor a replenishment opportunity occurrence is left. In this case, the class-2-backorder level at $t+L$ and the demand count at *t* are already resolved at their specified levels since there is nothing left to change the state of the system. □

Theorem 1 expresses the probabilities in (12) as a function of probabilities of the same type corresponding to smaller number of occurrences. These probabilities depend on *t*', which makes the direct application of the theorem impractical. However, if the replenishment opportunities and demand arrivals in $(t,t+L]$ are uniformly distributed, these probabilities are then independent of *t*'. In this case, the probabilities are then independent of the occurrence times and only depend on the number of occurrences.

Proposition 2 of Fadıloğlu and Bulut (2010a) also holds for the (Q,R) case once we replace the replenishment arrivals with replenishment opportunities. Thereby, it is reasonable to apply the same approximation with the lot-per-lot case and ignore the dependence between the class 2 backorders at *t* and the times of replenishment opportunities in $(t,t+L]$. Based on the proposition, one can conclude that the conditional arrival times are uniformly distributed. However, the probabilities in (11) not only specify the number of arrivals to occur, but also the number of class 2 backorders. The class 2 backorders at *t* are dependent on the demand times in $(t-L,t]$, which means that under the conditions that are given in (11), the replenishment opportunities in $(t,t+L]$ are no longer uniformly distributed. However, we claim that if that dependence is ignored and uniform distribution is assumed, the resulting analysis is an excellent approximation just as in the lot-per-lot case. We demonstrate the quality of the approximation in Section 4.

Once the uniformity on $(t,t+L]$ is assumed, we know that the remaining occurrences on $(t',t+L]$ are still uniformly distributed for any $t' \in (t,t+L]$. Thereby,

$$P(A(u)=r\,|\,b,y,z,t')=\frac{z}{y+z}\ ,\ P(A(u)=d\,|\,b,y,z,t')=\frac{y}{y+z},$$

and all the probabilities in (14) are independent of the first arrival time, *u*. When these are applied on Equation (14), we obtain

$$P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{matrix} X(t+L)=x_L, C(t'-L)=c, \\ B(t')=b, Y(t')=y, Z(t')=z \end{matrix}\right\}=$$

$$\frac{z}{z+y}\,P\left\{\begin{matrix} C(t)=c_L, \\ B(t+L)=b_L \end{matrix} \,\middle|\, \begin{matrix} X(t+L)=x_L, C(t'-L)=(c+1)\operatorname{mod} Q, \\ B(t')=\left(b-\left(Q-\left(\left(x+c-b\right)-(R+Q-K)\right)\right)^{+}\mathbf{1}\{c=Q-1\}\right)^{+}, \\ Y(t')=y, Z(t')=\left(z-1\right)^{+} \end{matrix}\right\}$$

$$+\frac{y}{z+y}\left(p_1 P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{matrix} X(t+L)=x_L, C(t'-L)=c, \\ B(t')=b, Y(t')=\left(y-1\right)^{+}, Z(t')=z \end{matrix}\right\}\right.$$

$$\left.+\,p_2 P\left\{C(t)=c_L, B(t+L)=b_L \,\middle|\, \begin{matrix} X(t+L)=x_L, C(t'-L)=c, \\ B(t')=b+\mathbf{1}\{x+c-b\geq R+Q-K\}, \\ Y(t')=\left(y-1\right)^{+}, Z(t')=z \end{matrix}\right\}\right). \tag{17}$$

Equation (17) is independent of *t*'. This independence provides us with a practical means to calculate probabilities of (12).

Based on (17), we propose a recursive procedure to compute the one-step transition probabilities of the embedded Markov chain for the inventory system under the (Q,R) ordering policy. Equation (15) determines the probabilities corresponding to the case at which neither a demand arrival nor a replenishment opportunity is left. Starting with this probability, all other probabilities of interest are computed recursively, invoking the interior equation (17) and increasing $Y(t')$ and then $Z(t')$ one by one. The procedure is completed by applying (13) and then (10) in order to obtain the one-step transition probabilities.

All the one-step transition probabilities can be generated with this procedure. The only problem is that the state space of the embedded Markov chain is infinite and it would take infinite amount of time to generate all the elements of the Markov transition matrix. Thereby, we have to find a way of working with a finite version, i.e., a truncation, of the original matrix. This problem is resolved in Section 3.

3. Steady-State Analysis

In this section, we present the steady-state analysis for the Markov chain obtained in Section 2. We consider a subset of the state space for which $0 \le X(t) \le D_{\max}$. $D_{\max}$ is the maximum number of outstanding replenishment orders we consider. The second dimension of the state space, $C(t-L)$, can take any value between 0 and $Q-1$, irrespective of the values of the other two state variables. Finally, since the third dimension of the state space, the number of class 2 backorders, is limited by the number of outstanding replenishment orders via (5) and (6), no truncation is needed for this dimension, i.e., $0 \le B(t) \le D_{\max} - R + K - 1$. One should note that when we truncate the state-space as described, we are effectively ignoring an infinite number of states whose total probability is equal to $P\{X(t) > D_{\max}\}$. Since *X(t)* has a Poisson distribution with parameter λL as stated in (7), we are ignoring the tail of Poisson distribution. Hence, as $D_{\max}$ increases, the probability of the ignored part of the state space goes to zero rapidly.

In order to perform the steady-state analysis, we need to generate the Markov transition matrix for the chain. Since we are unable to generate the full matrix, we generate a submatrix of the transition matrix, which corresponds to the states that are conserved by the truncation, i.e. that are not ignored. We call this submatrix **Q**. We consider a lexicographical ordering of these states in order to map the states to the columns and rows of the transition matrix. Then, one can map the state (x,c,b) to the $r(x,c,b)^{\text{th}}$ column and row in the transition matrix, where

$$r(x,c,b) = \sum_{x'=0}^{x-1}\sum_{c'=0}^{Q-1}\left(\left(x'+c'-(R+Q-K)\right)^{+}+1\right) + \sum_{c'=0}^{c-1}\left(\left(x+c'-(R+Q-K)\right)^{+}+1\right) + (b+1). \quad (18)$$

Note that $\left(\left(x'+c'-(R+Q-K)\right)^{+}+1\right)$ is the number of states for which $X(t) = x'$ and $C(t-L) = c'$. Thereby, the first –double– summation corresponds to the number of states that corresponds to all the states for which $0 \le X(t) \le x-1$. The second summation corresponds to the number of states for which $X(t) = x$ and $0 \le C(t-L) \le c-1$. Finally the third term in the right-hand-side of (18), corresponds to the number of states for which $X(t) = x$, $C(t-L) = c$, and $0 \le B(t) \le b$. Using (18), we can deduce that the total number of columns (or rows) in **Q** is $\sum_{x'=0}^{D_{\max}}\sum_{c'=0}^{Q-1}\left(\left(x'+c'-(R+Q-K)\right)^{+}+1\right)$.

Now, we present the recursive algorithm we use to generate the one-step transition probabilities corresponding to the conserved states:

Step 1: **Initialization (Boundary Condition)**

for x_L = 0 to $D_{\max}$

  for c_L = 0 to Q -1

    for b_L = 0 to $((x_L + c_L) - (R + Q - K))^+$

      for c = 0 to Q -1

        for b = 0 to $((x_L + c) - (R + Q - K))^+$

          Invoke (15).

Step 2: **Recursion**

for y = 0 to $D_{\max}$

  for z = $1\{y = 0\}$ to $D_{\max}$

    for x_L = 0 to $D_{\max}$

      for c_L = 0 to Q -1

        for b_L = 0 to $((x_L + c_L) - (R + Q - K))^+$

          for c = 0 to Q -1

            for b = 0 to $((x_L + z - y + c) - (R + Q - K))^+$

              Invoke (17).

Step 3: **Collection**

for x_0 = 0 to $D_{\max}$

  for c_0 = 0 to Q -1

    for b_0 = 0 to $((x_0 + c_0) - (R + Q - K))^+$

      for x_L = 0 to $D_{\max}$

        for c_L = 0 to Q -1

          for b_L = 0 to

            Invoke (13) to compute (11)

            Invoke (10) and obtain a one-step transition probability.

The algorithm consists of three steps. In the first step, the boundary condition given in (15) is applied for all feasible state variable values. The truncation operation determines the set of feasible state variable values. Thus, the probabilities of (12) corresponding to $y = z = 0$ are computed. In the second step, the recursive equation of (17) is used to obtain the probabilities corresponding to nonzero values of y and z by increasing them one at a time. Finally, the one-step transition probabilities are obtained in the third step by appropriately collecting the results of Step 2.

The number of conserved states increases with $D_{\max}$ and Q and is in the order of $QD_{\max}^2$. The recursive procedure computes the probability given in expression (12), for all feasible x_L, c_L, b_L, c, b, y, and z values. Each of x_L, b_L, b, y, and z is bounded by $D_{\max}$, since our truncation neglects the event that the number of demand or replenishment arrivals during lead-time is greater than $D_{\max}$. Moreover, c_L and c are bounded by Q. This means the computational complexity of the recursive algorithm is $O\left(Q^2 D_{\max}^5\right)$. Thus, as a result of a computational effort of $O\left(Q^2 D_{\max}^5\right)$, we are able to obtain the matrix **Q**, which is a finite submatrix of the original Markov transition matrix, which is infinite.

One could claim that once the truncation is performed, one does not have the original Markov chain, thereby an analysis based on this truncation would only be approximate. Although this claim is correct in the strictest sense of approximation, there is a theory in computational linear algebra, which states that one can get exact upper and lower bounds for steady-state probabilities corresponding to the nontruncated states by considering a truncated version of an irreducible Markov chain. The methodology we present is based on the theory in Courtois and Semal (1984).

It is observed that the upper and lower bounds for our problem converge together rapidly as $D_{\max}$ is increased. This is due to the additional structure our Markov chain possesses. Thus, we can get the steady-state probabilities with any desired accuracy.

In order to explain the procedure, let us call the Markov transition matrix **P**. The partitioning of **P** into submatrices corresponding to truncation states and to ignored states can be expressed as

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{A} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}. \tag{19}$$

One should note that **Q** is not a transition matrix, and thereby **Q** and (**I**-**Q**) are invertible given that **P** is a transition matrix corresponding to an irreducible chain. Let $\boldsymbol{\pi}$ be the steady-state probabilities vector corresponding to truncation states and $\boldsymbol{\pi}^{\mathbf{i}}$ the one corresponding to ignored states. Part of steady-state equations can be given as

$$\boldsymbol{\pi}\mathbf{Q}+\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}=\boldsymbol{\pi} \;\Rightarrow\; \boldsymbol{\pi}\left(\mathbf{I}-\mathbf{Q}\right)=\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B} \;\Rightarrow\; \boldsymbol{\pi}=\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}. \tag{20}$$

Equation (20) means that we could find steady-state probabilities corresponding to truncation states, if the vector $\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ were known. Based on this observation, one can construct another Markov transition matrix, $\tilde{\mathbf{P}}$, the truncated chain's transition matrix, by lumping all the ignored states into a single state, i.e.

$$\tilde{\mathbf{P}}=\begin{bmatrix}\mathbf{Q} & \mathbf{1}-\mathbf{Q1}\\ \mathbf{x} & 1-\mathbf{x1}\end{bmatrix} \quad \textit{where } \mathbf{1}=\begin{bmatrix}1 & \ldots & 1\end{bmatrix}^{T}. \tag{21}$$

The row vector $\mathbf{x}$ represents the probability vector at which the process corresponding to the truncated chain makes a transition from the lumped state to the other states.

Let us consider the matrix $\tilde{\mathbf{P}}$ when $\mathbf{x}=(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$. The vector $\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ can be interpreted as the probability vector at which the process makes a transition from the ignored states into truncated states. Then $(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ is the same transition probability under the condition that the process is in the ignored part of the state space. Therefore, the lumped state here mimics the dynamics of the ignored states. The steady-states probabilities corresponding to the matrix (21), will be the same with the ones of the original chain given that $\mathbf{x}=(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$. In order to show this, let us call the steady-state probabilities vector corresponding to truncation states under the new transition matrix $\tilde{\mathbf{P}}$ as $\tilde{\boldsymbol{\pi}}$ and the steady-state probability for the lumped state as π^{l}. Steady-state equations corresponding to truncation states are

$$\tilde{\boldsymbol{\pi}}\mathbf{Q}+\pi^{l}\left(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}\right)\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}=\tilde{\boldsymbol{\pi}} \;\Rightarrow\; \tilde{\boldsymbol{\pi}}\left(\mathbf{I}-\mathbf{Q}\right)=\pi^{l}\left(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}\right)\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B} \;\Rightarrow\; \tilde{\boldsymbol{\pi}}=\pi^{l}\left(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}\right)\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}. \tag{22}$$

The equation for the lumped state is not needed since it is linearly dependent with the equations in (22). This means that the steady-state probabilities can be obtained by setting π^{l} to 1, then finding the rest of the probabilities using (22), and finally normalizing them, i.e.

$$\tilde{\boldsymbol{\pi}}=\frac{1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}}{1+\left(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}\right)\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1}}\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}=\frac{1}{\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}+\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1}}\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}=\frac{1}{\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1}+\boldsymbol{\pi}\mathbf{1}}\boldsymbol{\pi}=\boldsymbol{\pi}. \tag{23}$$

This means that the both transition matrices, $\mathbf{P}$ and $\tilde{\mathbf{P}}$, yield the same steady-state probabilities for truncation states. Thereby, one could obtain certain steady-state probabilities of an infinite state space Markov chain, using a finite state space chain. The only problem is that in order to construct the finite chain, one needs to set vector $\mathbf{x}$ to $(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$, which requires the steady-state solution of the original. Since the solution is what we are after, this result does not have practical relevance. Yet based on this result, exact upper and lower bounds can be developed for steady-state probabilities of interest.

The idea behind the procedure yielding the bounds is as follows: Since the vector $(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ representing the transitions from the lumped state to the truncation states in $\tilde{\mathbf{P}}$ cannot be determined without having the steady-state solution of the original chain, one can try to see what happens if any vector $\mathbf{x}$ of the same dimension is used instead of $(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$. If we can find the steady-state probabilities vector solution set corresponding to all possible vectors, then the theory of Courtois and Semal (1984) states that the actual steady-state probabilities vector $\tilde{\boldsymbol{\pi}}$, which is equal to $\boldsymbol{\pi}$, has to be an element of this set. Moreover, it has been shown that this solution set forms a polyhedron, whose vertices are the solutions of the truncated chain with $\mathbf{x} = \mathbf{e_i}^T$ for all i's where $\mathbf{e_i}$'s are the standardized basis vectors ($\mathbf{e_i} = \begin{pmatrix}0 & \ldots & 0 & 1 & 0 & \ldots & 0\end{pmatrix}^T$). Since $(1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ is an element of the convex hull defined by the standardized basis vectors, it makes sense that the solution corresponding to $\mathbf{x} = (1/\boldsymbol{\pi}^{\mathbf{i}}\mathbf{1})\boldsymbol{\pi}^{\mathbf{i}}\mathbf{B}$ is an element of the convex hull defined by the solutions corresponding to the basis vectors.

One does not have to solve a different system of equations for every basis vector. In order to find all the solutions, it is enough to compute the inverse of (**I**-**Q**). Let $\tilde{\boldsymbol{\pi}}_{\mathbf{x}}$ be the solution of the truncated chain when the vector **x** is used. Then,

$$\tilde{\boldsymbol{\pi}}_{\mathbf{x}} = \pi^{l}\mathbf{x}\left(\mathbf{I}-\mathbf{Q}\right)^{-1} \quad \Rightarrow \tilde{\boldsymbol{\pi}}_{\mathbf{x}} = \frac{1}{1+\mathbf{x}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1}}\,\mathbf{x}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}, \tag{24}$$

using the same steps with the derivations given in (22) and (23). Basically $\tilde{\boldsymbol{\pi}}_{\mathbf{e_i}^T}$ can be obtained by taking the i^{th} row of $\left(\mathbf{I}-\mathbf{Q}\right)^{-1}$ and then applying normalization to it. The computational complexity of the inverse operation with *LU* factorization is $O(n^3)$ where n is the dimension of the matrix **Q**. Thence, since only one inverse operation is needed in order to apply our procedure, the computational complexity of the truncation algorithm is also $O(n^3)$, where n is the number of states that are not ignored in the truncation.

The bounds for individual steady-state probabilities can be obtained, once the polyhedron including the steady-state probability vector is known, by constructing a larger rectangular polyhedron covering the original polyhedron. The constructed polyhedron is defined by inequalities involving one dimension at a time. These bounds are given explicitly in Dayar and Stewart (1997). Let $z_{i,j}$ be the j^{th} element of the vector $\tilde{\boldsymbol{\pi}}_{\mathbf{e}_i^T}$, which is the steady-state probability corresponding to the j^{th} state in Markov chain defined by truncated chain with $\mathbf{x} = \mathbf{e_i}^T$. Then the upper and the lower bounds for the steady-state probability of state j are

$$\xi_j^{\inf} = \max\left\{\min_i\left(z_{i,j}\right);1-\sum_{k\neq j}\max_i\left(z_{i,k}\right)\right\}, \tag{25}$$

$$\xi_j^{\sup} = \min\left\{\max_i\left(z_{i,j}\right);1-\sum_{k\neq j}\min_i\left(z_{i,k}\right)\right\}. \tag{26}$$

In our problem, the size of **Q** is determined by the maximum demand considered during lead-time, $D_{\max}$. Since the dimension of **Q** is equal to the number of states consider the number n is in the order of $QD_{\max}^2$. This means the algorithm providing bounds on the steady-state probabilities has a computational complexity of $O\left(Q^3D_{\max}^6\right)$. The recursive algorithm generating the matrix **Q**, has a computational complexity of $O\left(Q^2D_{\max}^5\right)$ as we have discussed before. Thus, the computational complexity of the whole procedure we suggest is $O\left(Q^3D_{\max}^6\right)$. This means that as $D_{\max}$ increases the computation times increase accordingly. Yet, as $D_{\max}$ increases, a greater portion of the system's dynamics is included in the matrix **Q**, which means the quality of the bounds provided by our algorithm increases, i.e., bounds become tighter. We know that, when we set $D_{\max}$, we are effectively ignoring those states whose total steady-state probability is

$$P\left\{X(t) > D_{\max}\right\} = \sum_{x=D_{\max}+1}^{\infty} e^{-\lambda L}\frac{\left(\lambda L\right)^x}{x!}. \tag{27}$$

Since the Poisson probability mass function tends to zero rapidly beyond its mean value, one should select a $D_{\max}$ that is larger than the mean demand during lead-time, and increase it until the bounds are tight enough to yield the desired accuracy.

4. Numerical Study

In order to clearly demonstrate the tradeoff between the computation times and the quality of bounds, we provide the results of a numerical experiment. The algorithm described in this paper was coded in MATLAB™ computing language, then converted to C using MATLAB C Compiler, and was run on a PC with an AMD Athlon processor of 2.5 GHz clock speed and 1 GB of RAM.

We consider an inventory system of the type described at the beginning of Section 2, with the following parameters: $Q = 3$, $R = 3$, $K = 2$, $\lambda_1 = 1.5$, $\lambda_2 = 1.5$, $L = 1$. In Table 1, we provide results of our algorithm for the described system. We start our experiment with $D_{\max} = 3$, since the expected total demand during lead-time is 3. The tightness of the bounds is measured by the maximum bound gap, which is $\max_j \left\{ \xi_j^{\sup} - \xi_j^{\min} \right\}$. In order to see how the performance measures of interest are affected, estimates for fill rates of the first and second classes (β_1, β_2) are also reported in Table 1. Since the exact steady-state probabilities are not known, but upper and lower bounds are, the steady-state probabilities are assumed to be the midpoint in the interval. The probabilities ignored by the truncation are assumed to be zero. Hence, using PASTA property for Poisson arrivals we obtain

$$\hat{\beta}_1 = \sum_{x=0}^{D_{\max}} \sum_{c=0}^{Q-1} \sum_{b=0}^{(x+c-(R+Q-K))^+} 1\{x+c-b < R+Q\} \hat{P}\left\{ X = x, C = c, B = b \right\} \tag{28}$$

$$\hat{\beta}_2 = \sum_{x=0}^{D_{\max}} \sum_{c=0}^{Q-1} \sum_{b=0}^{(x+c-(R+Q-K))^+} 1\{x+c-b < R+Q-K\} \hat{P}\left\{ X = x, C = c, B = b \right\} \tag{29}$$

where $\hat{P}\left\{ X = x, C = c, B = b \right\} = \left(\xi_{r(x,c,b)}^{\sup} + \xi_{r(x,c,b)}^{\inf} \right) / 2$.

It should be also noted that there is no need to calculate the fill rate of the second class via our procedure. The exact value is provided in Deshpande et al. (2003). Nevertheless, we also provide our estimate for β_2 in order to demonstrate that our estimate converges to the theoretical value.

It is clear that maximum bound gap decreases uniformly as $D_{\max}$ increases. As the bound gaps decrease, the error in our steady-state probability estimates also decreases, which in turn causes the performance measure estimates to converge. In the experimental setting the average number of outstanding replenishment orders is 3. For systems with smaller λL's, convergence occurs at smaller values of $D_{\max}$.

Table 1 A Numerical Experiment (Q = 3, R = 3, K = 2, λ_1 = 1.5, λ_2 = 1.5, L = 1)

D_{max}	State Space Size	Computation Time (sec)	Maximum Bound Gap	$\hat{\beta}_1$	$\hat{\beta}_2$
3	13	0.047	3.3e-1	0.647	0.436
4	19	0.078	1.9e-1	0.782	0.396
5	28	0.204	9.4e-2	0.827	0.349
6	40	0.453	4.0e-2	0.836	0.338
7	55	0.984	1.4e-2	0.856	0.358
8	73	1.891	4.6e-3	0.885	0.386
9	94	3.469	1.4e-3	0.909	0.407
10	118	5.953	3.6e-4	0.922	0.417
11	145	9.406	9.0e-5	0.927	0.421
12	175	14.63	2.0e-5	0.929	0.423
13	208	21.92	4.1e-6	0.930	0.423
14	244	31.81	8.2e-7	0.930	0.423

In order to demonstrate the quality of the approximation involved in our methodology, we generated 36 system settings varying in policy parameters and arrival rates of the customer classes. In order to obtain the correct service levels for the system, we developed a simulation program in C. This program was run for 6 million customer arrivals in order to make sure that the service measures obtained were exact to the third significant digit. We observed that the estimates given by our algorithm are stable for different $D_{\max}$ values depending on the expected number of arrivals during lead-time (λL) of the experiment. For low (λL = 1), medium (λL = 3), and high (λL = 6) arrivals, we used $D_{\max}$ values of 7, 13, and 18, respectively. The results for the considered system settings are given in Table 2. The largest deviation of our algorithm from the exact values is 0.004, which is observed in the class 1 fill rate estimate when Q = 6, R = 3, and $\lambda L = \lambda_1 L + \lambda_2 L = 6$ (at both values of K). The average absolute deviation from the simulation estimates across the 36 settings is 0.001. This shows that our algorithm results in excellent fill rate estimates. Hence, the approximation due to ignoring the dependence of the replenishment times with the class 2 backorder –discussed in Section 2– does not cause any significant deviation from the true service levels.

Table 2 Comparison of Results with Simulation

			Our Algorithm				Deshpande's	Simulation	
			SS Size	Time (sec)	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_1$	$\hat{\beta}_1$	$\hat{\beta}_2$
$(Q = 3, R = 3, K = 2)$	$\lambda = 1$	$p_1 =$ 0.25			0.999	0.879	0.999	0.999	0.879
		0.50	55	1.1	0.998	0.879	0.998	0.998	0.879
		0.75			0.995	0.879	0.995	0.995	0.879
	$\lambda = 3$	$p_1 =$ 0.25			0.978	0.423	0.971	0.978	0.423
		0.50	208	22	0.930	0.423	0.912	0.930	0.423
		0.75			0.867	0.423	0.850	0.867	0.423
	$\lambda = 6$	$p_1 =$ 0.25			0.915	0.077	0.803	0.918	0.077
		0.50	418	119	0.759	0.077	0.557	0.759	0.077
		0.75			0.564	0.077	0.393	0.561	0.077
$(Q = 3, R = 3, K = 4)$	$\lambda = 1$	$p_1 =$ 0.25			1.000	0.368	1.000	1.000	0.368
		0.50	88	2.4	0.999	0.368	0.999	0.999	0.368
		0.75			0.997	0.368	0.997	0.997	0.368
	$\lambda = 3$	$p_1 =$ 0.25			0.998	0.083	0.996	0.998	0.083
		0.50	277	34	0.980	0.083	0.964	0.980	0.083
		0.75			0.925	0.083	0.892	0.924	0.083
	$\lambda = 6$	$p_1 =$ 0.25			0.990	0.007	0.956	0.993	0.007
		0.50	517	154	0.937	0.007	0.743	0.938	0.007
		0.75			0.789	0.007	0.486	0.786	0.007
$(Q = 6, R = 3, K = 3)$	$\lambda = 1$	$p_1 =$ 0.25			1.000	0.833	1.000	1.000	0.833
		0.50	104	3.3	0.999	0.833	0.999	0.999	0.833
		0.75			0.998	0.833	0.998	0.998	0.833
	$\lambda = 3$	$p_1 =$ 0.25			0.994	0.508	0.994	0.994	0.508
		0.50	392	75	0.971	0.508	0.970	0.971	0.508
		0.75			0.933	0.508	0.932	0.934	0.508
	$\lambda = 6$	$p_1 =$ 0.25			0.956	0.160	0.942	0.960	0.161
		0.50	797	399	0.838	0.160	0.783	0.841	0.161
		0.75			0.692	0.160	0.629	0.693	0.161
$(Q = 6, R = 3, K = 6)$	$\lambda = 1$	$p_1 =$ 0.25			1.000	0.337	1.000	1.000	0.337
		0.50	202	11	1.000	0.337	1.000	1.000	0.337
		0.75			0.999	0.337	0.999	0.999	0.337
	$\lambda = 3$	$p_1 =$ 0.25			0.999	0.112	1.000	1.000	0.112
		0.50	598	152	0.994	0.112	0.993	0.994	0.112
		0.75			0.967	0.112	0.959	0.968	0.112
	$\lambda = 6$	$p_1 =$ 0.25			0.994	0.014	0.995	0.998	0.014
		0.50	1093	691	0.967	0.014	0.915	0.971	0.014
		0.75			0.869	0.014	0.724	0.872	0.014

In Table 2, we also report the results of the analysis provided by Deshpande et al. (2003). This in effect compares our results with the ones of Arslan et al. (2007) since their results are the same as those of Deshpande et al. (2003). We do not report their results for class 2 fill rate since they are exact just like our results are. For Deshpande et al.'s method, the largest deviation from the exact result is 0.3 and the average absolute deviation is 0.041, which is 41 times greater than ours.

Although our approach yields significantly more accurate results, one has to note that the computation effort required by our algorithm is significantly greater as well. The largest computation time reported in our study is approximately 11 minutes. Of course for larger expected lead-time demands and lot sizes, the computation times increases. For certain extreme settings, the application of our algorithm may be computationally prohibitive. In such settings, some other approximations still have use. It would be interesting and useful to develop a faster approximation algorithm based on the underlying ideas for our methodology.

5. Conclusions and Extensions

In this paper, we provide a new method for the analysis of continuous-review inventory systems with backordering under (Q,R) ordering policy and static rationing with priority clearing. Our study is an extension of the study of Fadıloğlu and Bulut (2010a) in which the authors only consider the special case where $Q=1$. Our method culminates in an algorithm to compute the steady-state distribution for the inventory system. Although there are many approximate results in the literature, there was no method capturing the priority clearing dynamics for batch ordering systems up to this point. This constitutes the main contribution of the paper.

Our methodology points out that the dynamics of the rationing policy not only depends on the number of arrivals of different types but on their relative ordering as well. The effect of the relative order is quite complex as can be observed in the recursive procedure we propose. Thus, the problem we are attacking is combinatorial in nature. This explains why the problem resisted many attempts for its solution.

Our method is based on the observation that continuous-review inventory systems with backordering evolve according to a Markov chain at multiples of its lead-time. This means that when the system is sampled at multiples of lead-time, a discrete-time Markov chain is obtained. Moreover, the steady-state probabilities of this embedded Markov chain at hand are also valid for the underlying continuous-time inventory model. We provide a recursive algorithm to compute the probabilities defining the embedded chain. We note that since the state space of the embedded chain is infinite, one would need infinite amount of time to generate all the one-step transition probabilities, and then to obtain the steady-state probabilities. Then, we show that one can obtain upper and lower bounds for steady-state probabilities of certain states of a Markov chain using a truncated version of the chain. We explain how the quality of these bounds increase as the number of states conserved by the truncation is augmented. Finally, we show how the bounds can be used to obtain steady state probabilities of interest with desired accuracy

The scope of the method introduced in this paper can easily be extended. For example, it is straightforward to generalize the method to more than two priority classes. The state space dimension would have to increase by one for each additional class. But, that would of course come at a computation cost. Thus, it would be worthwhile to develop a faster approximate method to generate the one-step transition probabilities. This would considerably enhance the applicability of our algorithm. We believe that a computer program that implements the algorithm for any number of priority classes would be a handy tool. The described tool could also have policy optimization capability. To embed our algorithm in an optimization model to find the optimal rationing policy as done by Deshpande et al. (2003) and Arslan et al. (2007) is a straightforward addition to methodology presented in this paper. The optimization would be based on a search algorithm.

Acknowledgments

Special thanks to Tuğrul Dayar of Bilkent University Computer Science Department, who guided us in the field of computational linear algebra.

References

Arslan, H., S.C. Graves, T. Roemer. 2007. A single-product inventory model for multiple demand classes. *Management Science* **53** 1486-1500.

Bulut, O., M.M. Fadıloğlu. 2011. Production control and stock rationing for a make-to-stock system with parallel production channels. *IIE Transactions* forthcoming.

Courtois, P.-J., P. Semal. 1984. Bounds for the positive eigenvectors of nonnegative matrices and for their approximations by decomposition. *Journal of the Association for Computing Machinery* **31** 826-838.

Dayar, T., W.J. Stewart. 1997. Quasi lumpability, lower-bounding coupling matrices, and nearly completely decomposable Markov chains. *Siam J. Matrix Anal. Appl.* **18** 482-498.

Dekker, R., M.J. Kleijn, P.J. de Rooij. 1998. A spare parts stocking policy based on equipment criticality. *Int. J. Production Economics* **56-57** 69-77.

Dekker, R., R.M. Hill, M.j. Kleijn, R.H. Teunter. 2002. On the (S -1, S) lost sales inventory model with priority demand classes. *Naval Research Logistics* **49** 593-610.

Deshpande, V., M.A. Cohen, K. Donohue. 2003. A threshold inventory rationing policy for service-differentiated demand classes. *Management Science* **49** 683-703.

Fadıloğlu, M.M., Bulut, O. (2010a) An embedded Markov chain approach to stock rationing. *Operations Research Letters* **38** 510-515.

Fadıloğlu, M.M., Bulut, O. (2010b) A dynamic rationing policy for continuous-review inventory systems. *European Journal of Operational Research* **202** 675-685.

Gayon, J.P., Vericourt, F.D., and Karaesmen, F. (2009) Stock rationing in an $M/E_r/1$ multi-class make-to-stock queue with backorders. *IIE Transactions* **41** 1096-1109.

Ha, A.Y. 1997a. Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science* **43** 1093-1103.

Ha, A.Y. 1997b. Stock rationing policy for a make-to-stock production system with two priority classes and backordering. *Naval Research Logistics* **44** 457-472.

Ha, A.Y. 2000. Stock rationing in an $M/E_k/1$ make-to-stock queue. *Management Science* **46** 77-87.

Hadley, G., T. Whitin. 1963. *Analysis of Inventory Systems*. Prentice-Hall, Englewood Cliffs, NJ.

Kocaga, Y.L., A. Sen. 2007. Spare parts inventory management with demand lead-times and rationing. *IIE Transactions* **39** 879-898

Mollering, K.T., U.W. Thonemann. 2010. An optimal constant level rationing policy under service level constraints. *OR Spectrum* **32** 319-341.

Nahmias, S., W. Demmy. 1981. Operating characteristics of an inventory system with rationing. *Management Science* **27** 1236-1245.

Stidham, S. 1974. Stochastic clearing systems. *Stochastic and their applications* **2** 85-113.

Teunter, H.T., W.K.K. Haneveld. 2008. Dynamic inventory rationing strategies for inventory systems with two demand classes, Poisson demand and backordering. *European Journal of Operational Research* **190** 156–178.

Veinott, A.F. 1965. Optimal policy in a dynamic, single product, non-stationary inventory model with several demand classes. *Operations Research* **13** 761-778.

de Vericourt, F., F. Karaesmen, Y. Dallery. 2002. Optimal stock allocation for a capacitated supply system. *Management Science* **48** 1486-1501.