ON THE PROPERTIES OF MARKOVIAN MODELS OF PRODUCTION LINES

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Abstract: In this paper, we present a spectral theory pertaining to Quasi-Birth-Death Processes (QBDP). The QBDP, which is a generalization of the Birth-Death Process, is a powerful tool that can be utilized in modeling many stochastic phenomena including production lines [4]. Our theory is based on the application of a matrix polynomial method to obtain the steady-state probabilities in state-homogeneous finite-state QBDPs. We present different properties relating the quantities that arise in the solution procedure. These properties construct a unified theory on the spectral properties of QBDPs furnishing a formal framework to embody much of the previous work. This framework carries the prospect of furthering our understanding of the behavior the modeled systems manifest. We demonstrate how this framework can be exploited to analyse a well-known class of models, models of production lines, which have been studied by many scholars due to their importance in modern production systems. This approach leads to new insights about the balance of production lines.

1 Introduction and Past Work

In this paper, we first present a general theory for the spectral properties of state-homogeneous finite-state Quasi-Birth-Death Processes (QPDP). The spectral theory depicted here is then applied to models of production lines, which have been studied extensively by many scholars due to the importance of the subject [1].

The study of QBDPs was initiated by R. V. Evans [2] and the Ph.D. thesis of V. Wallace [8]. Wallace was the one to coin the term *quasi-birth-death processes*. The most detailed discussion of QBDPs is done by Neuts [6]. In his book, Neuts studies QBDPs as Markov chains using matrix geometric invariant vectors.

Although much previous research has been done on the subject of QBDPs, matrix polynomial approaches have only been recently applied. Consequently, the spectral properties of these processes, which are of fundamental importance in the application of matrix polynomial methods, have not been thoroughly investigated up to this point. This paper develops the theory, which is to constitute the general foundation for the application of matrix polynomial methods, and demonstrates how it can be applied to models of production lines.

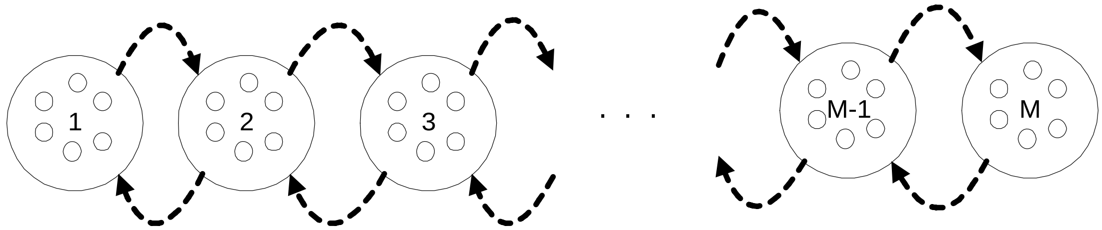

Figure 1 Representation of a QBDP (The dashed lines represent a collection of transitions originating from a state in one group and terminate in a state in another group.)

It is known from the work of Yeralan and Muth [9] that a state-homogenous finite-state QBDP gives rise to a transition rate matrix having a nested block tridiagonal structure.

$$R = \begin{bmatrix} B_0 & C & & & & & \\ A & B & C & & & & \\ & A & B & C & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & A & B & C \\ & & & & & A & B_M \end{bmatrix} \quad (1)$$

A method needs to be devised in order to take advantage of this structure. And here we argue that the use of matrix polynomial technique is quite propitious. That was first proposed by Tan and Yeralan [7,10].

2 The Matrix Polynomial Solution Procedure

The steady-state probability vector, π–towards which our effort is geared, –is known to satisfy the following equations.

$$\pi R = \underline{0} \quad (2)$$

$$\pi \underline{1}^T = 1 \quad (3)$$

where R is the transition rate matrix, $\underline{1}$ = (1,1,…,1), and $\underline{0}$ = (0,0,…,0). As we have already stated this probability vector is unique provided that R is irreducible. Any vector g satisfying the equation (2) is called a nonnormalized steady-state probability vector. Let (g_0,g_1, …,g_M) be a partitioning of the vector g in such a way that the size of each partition matches the dimension of the corresponding partition in R as manifested in equation (1). Thus we can state

$$gR = \underline{0} \quad (4)$$

Then using the partition notation we can rewrite equation (4).

$$g_0 B_0 + g_1 A = \underline{0} \tag{5}$$

$$g_{i-1} C + g_i B + g_{i+1} A = \underline{0} \quad for\ i = 1,2,\ldots, M-1 \tag{6}$$

$$g_{M-1} C + g_M B_M = \underline{0} \tag{7}$$

Equations (5) and (7) are named as the boundary equations and equation (6) as the interior equation. The interior equation is actually repeated M-1 times and can actually be classified as a difference equation. It has been shown by Gohberg, Lancester, and Rodman [5] that this matrix difference equation has solutions in the following form:

$$g_i = \lambda^i e \quad for\ i = 0,1,2,\ldots, M \tag{8}$$

where λ is scalar and e is a vector of the same dimension as g_i. In order to find all solutions of this family we substitute the proposed solution into the equation (6) and thereby obtain

$$\lambda^2 eA + \lambda eB + eC = \underline{0} \tag{9}$$

Now, we define the matrix polynomial $L(\lambda)$ in order to formalize our problem within the framework of matrix polynomial theory:

$$L(\lambda) = \lambda^2 A + \lambda B + C \tag{10}$$

Consequently, our goal–finding all vectors g satisfying equation (6)–can be reformulated as finding the eigenvalue-eigenvector pairs that belongs to the matrix polynomial $L(\lambda)$. We can also express the equation (9) using the new notation as

$$eL(\lambda) = \underline{0} \tag{11}$$

The set of all eigenvalue-eigenvector pairs (λ_i, e_i) of $L(\lambda)$ yields us a set of linearly independent solution vectors which spans the general solution space of the equation (6). The eigenvalues are the roots of the characteristic polynomial $\det(L(\lambda)) = 0$. One can readily observe this characteristic equation is of degree 2n and thereby has 2n solutions where n is the size of one dimension for each partition of R.

Let ρ be the number of the finite genuine and ρ_∞ the number of genuine eigenvalues at infinity. Let the series $(\lambda_j / j=1,2,\ldots,\rho+\rho_\infty)$ be a non-decreasing sequence of those eigenvalues with the eigenvectors at infinity at the end of the sequence. Then, the steady-state vector partitions are

$$g_i = \sum_{j=1}^{\rho} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} \lambda_j^l e_{(j,k-i+l)} + \delta(M,i) \sum_{j=\rho}^{\rho+\rho_\infty} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} e_{(j,k-i+l)} \quad \text{for } i=1,2,\ldots,M \tag{12}$$

where $s(\lambda_j)$ is the length of the generalized eigenvector cycle, or in other words, the size of the Jordan canonical form corresponding to the genuine eigenvector, λ_j; $e_{(j,k)}$ is the k^{th} generalized eigenvector of the generalized eigenvector cycle corresponding to the eigenvector, λ_j; and $w_{(j,k)}$ is the weight that corresponds to the k^{th} generalized eigenvector of the generalized eigenvector cycle corresponding to the eigenvector, λ_j.

Let $Com^{i}_{(i,j)}$ be the i[th] element of the collection that is the component that corresponds to k[th] generalized eigenvector of the j[th] eigenvalue, defined as

$$Com^{i}_{(j,k)} = \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} \lambda^{l}_{j} e_{(j,k-i+l)} \quad \text{, then,} \tag{13}$$

$$g_i = \sum_{j=1}^{\rho} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} Com^{\,i}_{(j,k)} + \delta(M,i) \sum_{j=\rho}^{\rho+\rho_\infty} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} Com^{\,M}_{(j,k)} \,. \tag{14}$$

Here, the word *component* is used for each expression that corresponds to a term starting with a $w_{(j,k)}$ in the equation (12). Thus, each component is a solution block corresponding to a given genuine or generalized eigenvector. At total there are 2n components. This is important since it is essential for the freedom of setting each $w_{(j,k)}$ arbitrarily, or more correctly, independently from the internal equations.

3 Spectral Properties of State-Homogeneous Finite-State Quasi-Birth-Death Processes

In this section we are going to delve in the spectral properties state-homogeneous finite-state QBDPs manifest. This section will directly be founded on the solution procedure developed in Section 2. We will omit the proofs of the presented theorems. Interested readers may find this development in another paper by the same authors [3].

Theorem 1: $\lambda_j e_{(j,1)} A \underline{1}^T = e_{(j,1)} C \underline{1}^T$ (15)

for all eigenvalues λ_j different from one.

The equation (15) is referred as a Balance Equation in Component Form (BECF) by Tan [6] who first proposed and proved it. Theorem 1 states that BECF hold for all eigenvalue-eigenvector pairs satisfying the matrix polynomial equation (11) except for those with eigenvalue one. One should notice that Theorem 1 does not state anything about the generalized eigenvectors that may be present in the solution due to multiplicity in the roots of characteristic equation.

Theorem 2: $e_{(j,k)}(\lambda_j A - C)\underline{1}^T = -e_{(j,k-1)} A \underline{1}^T$ (16)

for all eigenvalues λ_j different from one and k=2,3,... $s(\lambda_j)$ where $s(\lambda_j)$ is the length of the generalized eigenvector cycle, or in other words, the size of the Jordan canonical form corresponding to the eigenvector, λ_j.

We refer to equation (16) as the Raw Balance Equations in the Component Form (RBECF). These equations are for the generalized eigenvectors. Each expression depends on the previous element of the cycle. Yet, the next theorem shows that this expression still yields a balance component by component. The impurity introduced by the left hand-side of (16) is actually essential for this occurrence since each *component* also includes the previous eigenvectors of the same cycle. This fact is well manifested in the equation (12).

Theorem 3: $Com^{i}_{(j,k)}C\underline{1}^{T} = Com^{i+1}_{(j,k)}A\underline{1}^{T}$ (17)

for all (j,k) pair corresponding to an generalized eigenvector for which $\lambda_j \neq 1$ and for i = 0, 1, 3, . . ., M-1

We refer to equation (17) as the Generalized Balance Equations in Component Form (GBECF). Theorem 1 is a special case of the theorem. Yet this one is correct also for the generalized eigenvectors. This fact justifies name selected for it.

Theorem 4: There is one Jordan block that corresponds to the roots of the characteristic equation at one. That is, all roots at one give rise to a single cycle of generalized eigenvectors.

Theorem 5: The eigenvalue, $\lambda_j = 1$, has multiplicity greater than two if and only if $e_{(j,1)}A\underline{1}^{T} = e_{(j,1)}C\underline{1}^{T}$.

We refer to Theorem 5 as the Non-Balance at Unity Theorem (NBUT). This is the equivalent of BECF of Theorem 1 for the case of the eigenvalue at one. Yet, we observe that when the multiplicity of the eigenvalue at one is one, there is no balance of flow in component form for the *component* that corresponds to this eigenvalue.

Theorem 6: For m≥3, the eigenvalue, $\lambda_j = 1$, has multiplicity greater than m if and only if $e_{(j,1)}A\underline{1}^{T} = e_{(j,1)}C\underline{1}^{T}$ and $e_{(j,k)}(A-C)\underline{1}^{T} = -e_{(j,k-1)}(A)\underline{1}^{T}$ for 2≤ k ≤ m-1.

We refer to Theorem 6 as the Deficient Raw Balance at Unity Theorem (DRBUT). This is the equivalent of RBECF of Theorem 2 for the case of the eigenvalue at one. Yet, the equation that would cause the balance of flow in component form for the *components* corresponding to the last element of the cycle of eigenvectors that belongs to the eigenvector at one, never holds. For the previous elements of the same cycle, equalities, which will cause the balance of flow in component form for the *components* corresponding to these elements, hold.

Theorem 7: For $\lambda_j = 1$, the following statements are true:

$$Com^{i}_{(j,k)}C\underline{1}^{T} = Com^{i+1}_{(j,k)}A\underline{1}^{T} \text{ for } i = 0,1,2,...,M-1 \text{ and } k = 1,2,...,s(\lambda_j)-1$$

$$Com^{i}_{(j,s(\lambda_j))}C\underline{1}^{T} \neq Com^{i+1}_{(j,s(\lambda_j))}A\underline{1}^{T} \text{ for } i = 0,1,2,...,M-1$$

We refer to Theorem 7 as the Deficient Generalized Balance at Unity theorem (DGBUT). This is the equivalent of GBECF of Theorem 3 for the case of the eigenvalue at one. Yet, there is no balance of flow in component form for the *components* corresponding to the last element of the cycle of eigenvectors that belongs to the eigenvector at one. For the previous elements of the same cycle, the balance of flow in component form for the components corresponding to these elements exists.

Theorem 8: The coefficient in the general solution, corresponding to the last element of the eigenvector cycle that corresponds to, $\lambda_j = 1$, is always zero.

4 General Implications of the Spectral Theory

The quantities of central importance in the matrix polynomial procedure are the eigenvalues of the characteristic equation for a given QBDP. In the previous section, we have demonstrated that for each modeled QBDP, one could easily associate a characteristic equation. Although the term characteristic equation belongs to the mathematical concept being employed, it is also quite befitting from a modeling perspective. The roots of the characteristic equation, the eigenvalues, determine the behavior of the solution, thereby the steady-state characteristics of the model.

The solution for a given QBDP model is always a linear combination of the *components* that we have defined by (13). Each *component* actually corresponds to a generalized eigenvector of an eigenvalue of the system. The closed-form expression for a component includes the value of the eigenvalue and the elements of the cycle of generalized eigenvectors from the first element to the given eigenvector. The general solution for a given QBDP is given in (14).

When we examine this solution structure we see that any *component* is a solution candidate. Furthermore the spectral theory shows us that each of these *components* acts like the solution on their own. That means, the elements forming the solution all have the properties of the full solution. This is demonstrated by the fact that all the *components* have a balance of flow property within themselves. Thus, the balance of flow property for the full solution is not a property that manifests itself only at that level, it is the consequence of the fact that each of the elements that form the solution exhibit it on their own.

Each *component* consists of two building blocks: the eigenvalue and the eigenvectors–just one if the eigenvalue has a simple eigenvector corresponding to it, otherwise the elements of the generalized eigenvector cycle. Yet, the eigenvalue is the more crucial block since it determines the behavior of the *component*. If the eigenvalue's norm is greater than one, the *component* that corresponds to the eigenvalue will become more and more pronounced for larger elements of the counting space. Complementarily, if the eigenvalue's norm is smaller than one, the *component* that corresponds to the eigenvalue will become less and less pronounced for larger elements of the counting space. If the eigenvalue is at one, the component will have equal contribution all over the counting set. We argue that the existence of an eigenvalue at one relates to a physical balance concept, which manifests itself in an interesting way in models of production lines. Furthermore, if the eigenvalue is complex, one would observe an oscillatory behavior in the *component* corresponding to it.

If we have the eigenvalues of the system at hand we can tell quite a bit about the possible behavior of the system. As in the control theory one could even try to make a root-locus diagram for the eigenvalues for design and sensitivity analysis purposes. If one knows how the system parameters change the eigenvalues of the system, one can use this information to perturb the system towards a desired behavior.

5 Application to Models of Production Lines

Another paper by the same authors [4] illustrates how Markovian models of production lines can be fit into the QBDP platform. All production lines with buffers are actually suitable to be modelled as a QBDP. One of the buffers is modeled as counting set of QBDP. The mentioned paper takes eight examples of Markovian production lines models and demonstrates how formulates them as QBDPs by specifying the submatrices seen in the transition rate matrix of (1).

In this section we are going to take a production line model and by applying the spectral theory we have stated in Section 3, we will attempt to get new insight on the Markovian model. We had done the same application to many models of production lines. Yet our purpose here is to illustrate this application on one model.

The model we are considering is exponential server with breakdown and repair. The reader who wants to scrutinize the retrieval of the submatrices we are going to merely state is referred to the aforementioned paper.

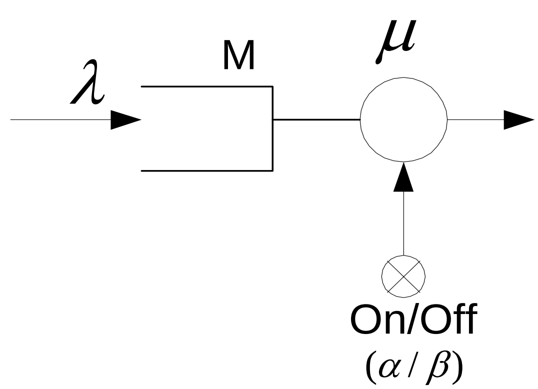

Figure 2 Exponential server with breakdown and repair and its relevant parameters

If the server is operating, it breaksdown with a time to breakdown distribution that is exponential with parameter α . Once the breakdown occurs, the repair starts and time to the end of repair is also exponentially distributed with parameter β . During the breakdown, the part that was being processed at the time breakdown occurred is not processed. Processing starts on the same part after the repair.

The submatrices defined in (1) are given below for this QBDP:

$$A=\begin{bmatrix}\mu & 0\\ 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}-(\alpha+\lambda+\mu) & \alpha\\ \beta & -(\beta+\lambda)\end{bmatrix},\quad C=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$$

$$B_0=\begin{bmatrix}-(\alpha+\lambda) & \alpha\\ \beta & -(\beta+\lambda)\end{bmatrix},\quad B_M=\begin{bmatrix}-(\alpha+\mu) & \alpha\\ \beta & -\beta\end{bmatrix}$$

The characteristic equation can be easily derived from the matrix polynomial

$$\begin{aligned}\det(L(x)) = &-\mu(\lambda+\beta)x^3+(2\lambda\mu+\alpha\lambda+\lambda^2+\beta\lambda+\beta\mu)x\\ &-(\alpha\lambda+\lambda\mu+\beta\lambda+2\lambda^2)x+\lambda^2\end{aligned} \tag{18}$$

One can express in closed-form all the eigenvalues of the system:

$$x_1 = \frac{\lambda}{2\mu}\left(\left(1+\frac{\alpha+\mu}{\beta+\lambda}\right) - \sqrt{\left(1+\frac{\alpha+\mu}{\beta+\lambda}\right)^2 - \frac{4\mu}{\beta+\lambda}}\right) \tag{19}$$

$$x_2 = \frac{\lambda}{2\mu}\left(\left(1+\frac{\alpha+\mu}{\beta+\lambda}\right) + \sqrt{\left(1+\frac{\alpha+\mu}{\beta+\lambda}\right)^2 - \frac{4\mu}{\beta+\lambda}}\right) \tag{20}$$

$$x_3 = 1 \quad , \; x_4 = \infty$$

It can be easily shown all eigenvalues are positive and non-complex. Thus, no oscilatory behavior is present in the corresponding *components*. As one can see, the first two eigenvalues are more critical for the understanding of the system, since they depend on the system parameters. The eigenvalue at one will not affect the solution by Theorem 8.

All the parameters that we make use of here have the unit, 1/[Time Unit]. Therefore their actual values depend on the choice of time unit. Yet one can express the eigenvalues in terms of parameters that are independent of the choice of time unit, while decreasing the number of parameters by one. In Model 1, this is done by the introduction of ρ, the traffic intensity. For this model, the same parameter can also be employed but we need to define two more parameters:

$\gamma = \frac{\alpha}{\lambda}$ (Breakdown intensity) and $\delta = \frac{\beta}{\lambda}$ (Repair intensity)

These definitions are useful from an engineering point of view, since it relates the eigenvalues of the system, to physically meaningful quantities. Thence, they make the relation between the system assumptions and the eigenvalues, which determine the conduct of the system, more visible.

$$x_1 = \frac{\rho}{2}\left(\left(1+\frac{1+\rho\gamma}{\rho(1+\delta)}\right) - \sqrt{\left(1+\frac{1+\rho\gamma}{\rho(1+\delta)}\right)^2 - \frac{4}{\rho(1+\delta)}}\right) \tag{21}$$

$$x_2 = \frac{\rho}{2}\left(\left(1+\frac{1+\rho\gamma}{\rho(1+\delta)}\right) + \sqrt{\left(1+\frac{1+\rho\gamma}{\rho(1+\delta)}\right)^2 - \frac{4}{\rho(1+\delta)}}\right) \tag{22}$$

Figure 3 clearly shows what happens as the traffic intensity changes. The first eigenvalue starts from a point under the unity level, and after crossing the unity level it tends to infinity. Whereas the second eigenvalue starts from zero and asymptotically approaches a level below the unity. The second eigenvalue never crosses the unity level. Both of them are monotonically increasing functions.

We can further support our observations by checking the limits of the eigenvalue functions:

$$\lim_{\rho \to 0^+} x_1 = 0 \text{ and } \lim_{\rho \to 0^+} x_2 = \frac{1}{1+\delta}$$

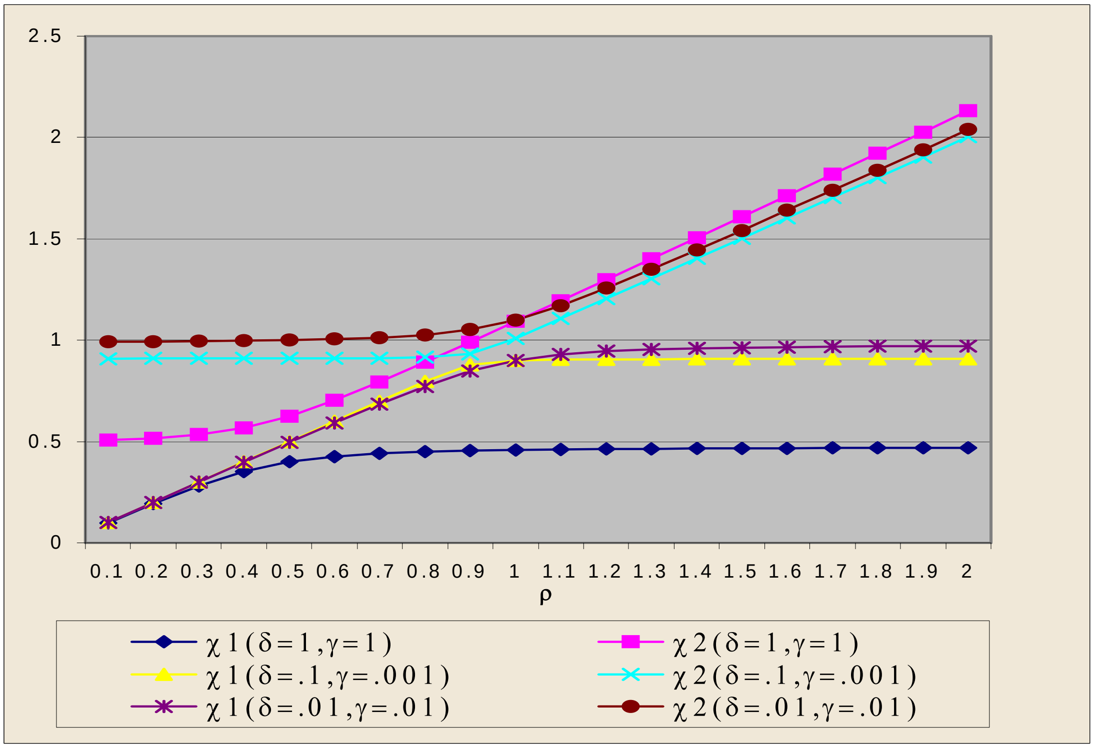

Figure 3 Eigenvalues drawn as a function of ρ, functions for different system parameters, γ and δ.

$\lim_{\rho \to \infty} x_2 = \infty$ (it tends to infinity like the function, $\left(1+\frac{\gamma}{(1+\delta)}\right)\rho$), and

$\lim_{\rho \to \infty} x_1 = \frac{1}{1+\delta+\gamma}$.

At this stage, we have been able to isolate two asymptotes for each eigenvalue. The first eigenvalue is asymptotic to the function $f(\rho) = \rho$ as it approaches to zero, and asymptotic to $f(\rho) = \frac{1}{1+\delta+\gamma}$ as it tends to infinity. Likewise, the second eigenvalue is asymptotic to $f(\rho) = \frac{1}{1+\delta}$ as it approaches to zero, and asymptotic to $f(\rho) = \left(1+\frac{\gamma}{1+\delta}\right)\rho$ as it tends to infinity. Moreover, it can be shown that the eigenvalue functions are monotonically increasing functions with respect to traffic intensity by checking their derivatives.

At this point, one can conclude that the both of the components are more biased towards the smaller buffer levels when the traffic intensity is relatively small. As the traffic intensity increases, this bias becomes less and less evident. At one point, the second eigenvalue transcends the unity. After this point, the second component is biased towards the larger buffer sizes. Yet, the first component is always more prominent at smaller buffer levels.

The point at which the second eigenvalue crosses the unity is also of utmost importance. At that point the second eigenvalue is at unity, which means that the component corresponding to it is balanced over the buffer levels. One can obtain the roots of the equation, $x_2 = 1$. The solution to this equation is

$$\rho = \frac{\delta}{\delta + \gamma} \tag{23}$$

As we can easily show, the effective average production rate of the system–adjusted for breakdowns–is $\mu \frac{\beta}{\alpha + \beta}$. The balance occurs when the average production rate is the same with the average arrival rate, λ. Thus for a balanced system one should have

$$\lambda = \mu \frac{\beta}{\alpha + \beta} \qquad (24))$$

This result is quite remarkable, since it is obvious that (23) is equivalent to (24). Thus, the balance conditions one derives for a balanced system through solely physical arguments, gives us an eigenvalue at one. This shows lucidly that the eigenvalues at one relate to the balance of system. And for more complicated problems, one can always check numerically if there are eigenvalues close enough to one.

One should observe that from a perspective of averages, only the proportion of parameters γ and δ matters. Yet, for the functional forms of the eigenvalues we see that the terms appear in such forms that they cannot be reduced to a proportion. Thus, even when the proportion of length of the broken periods to length of working period is kept constant, the system behavior changes. One needs to investigate the extent of this change, by examining at the eigenvalues for different system parameters having a constant proportion.

6 Spectral Properties of Markovian Models of Production Lines

In the previous section, we have demonstrated how the spectral theory we have derived can be applied to a given model of production lines. In our research we have exercised a similar application on many models of production lines. As a result, we were able to discern many common traits in the models of production lines.

Our first observation is the fact that all eigenvalues that pertain to models of production lines are positive and non-complex. This should be due to the special characteristics of the submatrices that are involved in QBDPs. It may be possible to prove this by using the properties of stochastic submatrices. This may be a worthwhile addition to our work.

An immediate consequence of the positive, non-complex nature of the eigenvalues is the fact that the *components* do not exhibit oscillatory behavior. That means, a *component*'s effect is either monotonically increasing or monotonically decreasing as the higher buffer levels are considered, or it is constant for all buffer levels. That means a *component* is either biased towards lower buffer levels or higher buffer levels.

Another interesting observation is that the eigenvalue functions never intersect each other, with the exception of the eigenvalue at the unity being intersected only once by only one of the eigenvalue functions. That means multiplicity only occurs at the unity, not for any other value.

Furthermore, the aforementioned intersection is of considerable interest, since it occurs when the system is physically balanced around the buffer. Here, we define a new balance concept with a buffer-centric approach. The system is balanced with respect to a buffer–according to our definition, –when the production rate of the line that is downstream with respect to the buffer is equal to the production rate of the line that is upstream with respect to the buffer, given that the buffer is taken out of the system and the system is divided to two parts.

Using the fact that the boundary equations do not effect the location of the balance point, we can conclude that the balance of a system that can be modeled by a QBDP is independent of the boundary behavior of the system. This idea can be used in finding the balance point for quite complicated systems with the help of numerical analysis tools.

The contribution of this different balance concept is important, because it describes a balanced probability distribution for a production line, whereas the old balance concept, which relates to the equality of stand-alone production rates of all servers, cannot guarantee such a distribution. This balance definition takes into consideration the starvation and the blocking in the system, which truly affects the balance in the system.

Another observation is that, it is always possible to define traffic intensity for any production line. As the line gets more complicated, one needs to introduce some other parameters. Yet, it is always of interest to look at the conduct of the eigenvalues as a function of the traffic intensity, since it provides a tool we can employ to compare different systems.

The eigenvalues at zero and infinity can be considered as correction masses that are employed for zero and full buffer levels. They strictly contribute to the steady-state probabilities at these points. They can be thus related to the phenomena such as starvation and blockage.

The knowledge of eigenvalues conduct can be very useful for making sensitivity analysis. One can check how much the systems behavior would change if the parameters were perturbed, by considering the locus of eigenvalues. This can be quite useful at the design of a production line.

Another fact that can be extremely useful, is the fact that the boundary equations can be used to manipulate the composition of final steady-state probabilities. We have already explained how each *component* is a solution candidate and the boundary equations determine how much of each component goes into the final result.

Yet, our experimentation with the models suggests that there is a strong coupling between the components. Even by applying great perturbations to the boundary equations, the amount of change in the composition of the final result is limited. But this can still be used to push the system in the desired direction to a certain extent. Moreover, this fact insinuates that the behavior of the system in the boundary is more critical than one may expect.

In retrospect, the spectral theory provided new tools that furthered our understanding of the behavior of the steady-state probabilities for models of production lines. These tools are beneficial for the sensitivity analysis of the models. They can be quite instrumental at the design of the production lines and at pushing them towards a more desirable conduct.

6 Conclusions

This work formalizes the matrix-polynomial approach for the analysis of QBDPs. We provide new properties relating the quantities of interest in the matrix-polynomial solution procedure. Most importantly, we unify the new and the previously stated properties in a formal framework. Furthermore, we demonstrate how this framework can be used to further the understanding of the probabilistic behavior of production lines that can be modeled using the QBDPs. Our approach illustrated here is yet applicable to all areas of stochastic modeling where QBDPs arise.

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