

## An EOQ Model with Multiple Suppliers and Random Capacity

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**ABSTRACT.** We consider an EOQ model with multiple suppliers that have random capacities which lead to uncertain yield in orders. A given order is fully received from a supplier if the order quantity is less than the supplier's capacity; otherwise, the quantity received is equal to the available capacity. The optimal order quantities for the suppliers can be obtained as the unique solution of an implicit set of equations in which the expected unsatisfied order is the same for each supplier. Further characterizations and properties are obtained for the uniform and exponential capacity cases with discussions on the issues related to diversification among suppliers.

**Keywords.** EOQ, random capacity, multiple suppliers, order diversification.

### 1. INTRODUCTION

Continuous-review inventory systems with random yield has been modelled in several different ways in the literature. The original idea behind yield randomness is due to the fact that the quantity received from the supplier may differ somewhat from the quantity ordered. As discussed in the review by Yano and Lee [1995], a common way to model yield uncertainty is to take the random yield  $Y_q$  "stochastically proportional" to the order quantity  $q$  so that  $Y_q = Uq$ . Here,  $U$  is a random variable which may represent, for example, the fraction of non-defective items. Earlier examples of these models can be found in Karlin [1958], Silver [1976] and Shih [1980].

Lee and Yano [1988] formulate the multistage serial production system with random yield and deterministic demand. Henig and Gerchak [1990] provide a comprehensive analysis of general periodic-review models with random yield in multi-periods and show the optimality of “nonorder-up-to” policies. Unfortunately, these policies are not as simple as the well-known base-stock and  $(s, S)$  policies. Under such policies, no order is given if inventory position is over a critical threshold, but the order quantity below this level does not necessarily bring the inventory position to a fixed base-stock level. Parlar and Berkin [1991] and Gürler and Parlar [1997] analyze the case where supply is available only during intervals of random length. Özekici and Parlar [1999] introduce the idea of a random environment which affects the demand, supply and all cost parameters. They show the optimality of environment-dependent base-stock and  $(s, S)$  policies when the supplier is unreliable.

Another approach in modelling random yield is to treat yield uncertainty as a consequence of random capacity. This may be due to unreliable machinery and unplanned maintenance in a production system or possibly finite availability of items in an inventory system. In these models, the quantity that is actually received is  $Y_q = \min\{q, A\}$  if  $q$  is the order quantity and  $A$  is the random capacity. Ciarallo et al. [1994], for example, propose a periodic-review production model with random capacity where the base stock policy is found to be optimal. Wang and Gerchak [1996a] further extend this model to allow random capacity and random yield simultaneously, i.e.,  $Y_q = U \min\{q, A\}$ . The structure of the optimal ordering policy in each period is similar to that of Henig and Gerchak [1990]; i.e., an order is given if the inventory level at the beginning of the period is below a critical point, otherwise no order is given. Güllü [1998] considers a model where the yield depends on the quantity present at the supplier in addition to the availability of the supplier. Erdem and Özekici [2002] analyze periodic models with random capacity in a random environment and show the optimality of environment-dependent base-stock policies when there is no fixed cost. In continuous-review environments, Wang and Gerchak [1996b] analyze the effects of variable capacity on optimal lot size and obtain optimality conditions for generally distributed capacity.

An important factor that is missing or neglected in the literature is the necessity to include multiple suppliers in random capacity models. Random supplier capacity directly implies that orders should be diversified to many suppliers in order to reduce the risk associated with insufficient capacity of the suppliers. In practice, the assumption that there is a single supplier is often false and unrealistic since variability of actual yield can be reduced through diversification of the risk by working with a number of suppliers. Extensive studies have been done on the random yield and random capacity models with a single supplier. On the other hand, substantially less effort has been spent on models with multiple suppliers due mainly to their apparent complexity. An original work is by Anupindi and Akella [1993] where they consider the single-period problem of ordering from two different suppliers with stochastically proportional yields. The optimal policy, which is determined by two critical order

points, is of the form: “order from both”, “order from the cheaper supplier” or “do not order”. The issue of order diversification is discussed by Erdem [1999] in a single-period model where there are two suppliers with random capacities to show that the total order quantity does not necessarily bring the inventory position to a base-stock level. Even with no fixed cost of ordering, the optimal policy may be rather complicated. In a continuous-review system, diversification under yield randomness was first analyzed in the EOQ context by Gerchak and Parlar [1990] where two suppliers with identical cost parameters and nonidentical stochastically proportional yields are considered. Parlar and Wang [1993] extended these results to supplier-specific unit cost of ordering and found the optimal order quantities explicitly. All these multiple supplier studies concentrate on two supplier models. The only other study that considers an unlimited number of suppliers like our study is by Fadılođlu et. al. [2005]. They analyze the multiple supplier binomial yield problem in an EOQ setting and show that diversification is not always preferable.

In this paper, we discuss issues related to random capacity and multiple suppliers in the well-known EOQ model. Our problem is similar to that of Parlar and Wang [1993] since we both discuss yield uncertainty and multiple suppliers in the EOQ model, but the yield structures are substantially different and the number of suppliers is not limited to two. Another closely related study is by Wang and Gerchak [1996b] where there is a single supplier with random capacity. We generalize these models to allow multiple nonidentical suppliers with random capacities in a continuous-review system.

The organization of the paper is as follows: In Section 2, we derive the general characterization for the optimal order quantities of the random capacity EOQ model with multiple suppliers. Sections 3 and 4 are devoted to the special cases of the uniform and exponential capacity suppliers where some interesting properties of the optimal solution are provided. In Section 5, the issues related to diversification among suppliers are discussed and results are demonstrated by some numerical illustrations. The reader should refer to the Appendix for the lengthy proofs.

## 2. EOQ MODEL WITH MULTIPLE SUPPLIERS

If it is certain that the suppliers deliver what is ordered, then it is surely most economical to work with the supplier who provides the product at the least cost for the desired quality level. However, this is often not the case in real life and it is a fact that usually companies prefer to work with more than one supplier. Under random supplier capacities, the retailer should order from a number of suppliers in order to diversify the risk associated with shortages.

In this section, we model the setting described above by assuming that there are  $n$  suppliers with constant lead times. Joint orders are given to  $n$  suppliers with independent random capacities  $\{A_i; i = 1, 2, \dots, n\}$  having distribution functions  $F_i$  and density  $f_i$ . Let  $q_i$  be the order quantity for the  $i$ th supplier, then the amount received from supplier  $i$  is  $Y_{q_i} = \min\{q_i, A_i\}$ . At the beginning of any order cycle,

if  $q_1, q_2, \dots, q_n$  are the order quantities for the  $n$  suppliers, then the total amount actually received from the joint order is  $Y_{q_1} + Y_{q_2} + \dots + Y_{q_n}$ . We assume that  $\bar{F}_i(x) > 0$  for all  $x \geq 0$  unless stated otherwise throughout the remainder of the paper. However we may relax this assumption and analyze the model when there is an upper bound  $a_i$  for the capacity of the supplier  $i$ , such that  $\bar{F}_i(x) = 0$  for all  $x \geq a_i$ . This is illustrated in the uniform capacities setting in Section 3.

Our model assumes that all the available suppliers are used. The joint order (setup) cost is  $K_n$ , the cost of giving an order to all of the  $n$  available suppliers. If which subset of the suppliers used is also a decision, then the order cost would be a function of the subset that would be the sum of a supplier specific (minor) setup and an order specific (major) setup. This situation is discussed and illustrated in Section 5.3.

The unit purchase cost is  $c$  and unit inventory holding cost is  $h$  per unit time. We suppose that all suppliers offer the same unit price since we want to focus on the effect of random capacity on the ordering policy.

Using renewal theory, the long run average total cost function is simply the ratio of expected total cost per cycle to expected cycle length. The required expression for the average cost as a function of the order quantities for the  $n$  suppliers is

$$\frac{K_n + cE[Y_{q_1} + Y_{q_2} + \dots + Y_{q_n}] + hE[(Y_{q_1} + Y_{q_2} + \dots + Y_{q_n})^2]/(2D)}{E[Y_{q_1} + Y_{q_2} + \dots + Y_{q_n}]/D}. \quad (1)$$

The numerator of (1) is the sum of joint ordering cost, expected cost of purchasing and the expected total inventory holding cost in a cycle. Now, we derive an expression for  $E[(Y_{q_1} + Y_{q_2} + \dots + Y_{q_n})^2]$  that will simplify (1). Since the random capacities are assumed to be independent, we have

$$E[(Y_{q_1} + Y_{q_2} + \dots + Y_{q_n})^2] = \sum_{i=1}^n E[Y_{q_i}^2] + \sum_{i,j=1, j \neq i}^n E[Y_{q_i}]E[Y_{q_j}]. \quad (2)$$

The average total cost expression in (1) can now be rewritten as

$$cD + \frac{K_n D + h \left[ \sum_{i=1}^n E[Y_{q_i}^2] + \sum_{i,j=1, j \neq i}^n E[Y_{q_i}]E[Y_{q_j}] \right] / 2}{\sum_{i=1}^n E[Y_{q_i}]}. \quad (3)$$

The total purchase cost  $cD$  can be disregarded in the optimization problem so that it now becomes  $\min_{q_1, q_2, \dots, q_n \geq 0} TC(q_1, q_2, \dots, q_n)$  where

$$TC(q_1, q_2, \dots, q_n) = \frac{K_n D + h \left[ \sum_{i=1}^n E[Y_{q_i}^2] + \sum_{i,j=1, j \neq i}^n E[Y_{q_i}]E[Y_{q_j}] \right] / 2}{\sum_{i=1}^n E[Y_{q_i}]}. \quad (4)$$

At this point, we need to provide the expressions for  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  in (4). We can compute  $E[Y_{q_i}]$  as

$$E[Y_{q_i}] = \int_0^{+\infty} \min\{q_i, y\} dF_i(y) = \int_0^{q_i} y dF_i(y) + q_i \bar{F}_i(q_i). \quad (5)$$

Note that  $E[Y_{q_i}] \leq E[A_i]$  with  $\lim_{q_i \rightarrow +\infty} E[Y_{q_i}] = E[A_i]$ . The derivative of  $E[Y_{q_i}]$  with respect to order quantity  $q_i$  is nonnegative since

$$\frac{dE[Y_{q_i}]}{dq_i} = \int_{q_i}^{+\infty} dF_i(y) = \bar{F}_i(q_i) \geq 0 \quad (6)$$

while the second derivative is nonpositive since

$$\frac{d^2E[Y_{q_i}]}{dq_i^2} = -f_i(q_i) \leq 0. \quad (7)$$

We conclude that  $E[Y_{q_i}]$  is a concave increasing function with  $E[Y_{q_i}]|_{q_i=0} = 0$  and it converges to  $E[A_i]$  as  $q_i$  increases. Similarly, we obtain  $E[Y_{q_i}^2]$  by using  $Y_{q_i} = \min\{q_i, A_i\}$  as

$$E[Y_{q_i}^2] = \int_0^{+\infty} \min\{q_i, y\}^2 dF_i(y) = \int_0^{q_i} y^2 dF_i(y) + q_i^2 \bar{F}_i(q_i). \quad (8)$$

Now,  $E[Y_{q_i}^2] \leq E[A_i^2]$  with  $\lim_{q_i \rightarrow +\infty} E[Y_{q_i}^2] = E[A_i^2]$ . Furthermore, we differentiate  $E[Y_{q_i}^2]$  in (8) with respect to  $q_i$  so that

$$\frac{dE[Y_{q_i}^2]}{dq_i} = 2q_i \int_{q_i}^{+\infty} dF_i(y) = 2q_i \bar{F}_i(q_i) \geq 0 \quad (9)$$

and find that  $E[Y_{q_i}^2]$  is also an increasing function with  $E[Y_{q_i}^2]|_{q_i=0} = 0$  and converging to  $E[A_i^2]$  as  $q_i$  increases. However, it is not necessarily concave.

We can now get into the details of solving our optimization problem to minimize  $TC(q_1, q_2, \dots, q_n)$ . The following result provides an interesting implicit characterization for the optimal solution of the random capacity EOQ model with multiple suppliers.

**Theorem 1.** *The optimal order quantities of the random capacity EOQ model with  $n$  suppliers are given as the unique nonnegative solution of the following set of equations*

$$2q_1 \sum_{i=1}^n E[Y_{q_i}] + 2 \sum_{i=2}^n \sum_{j=i}^n E[Y_{q_i}] E[Y_{q_j}] - \sum_{i=1}^n E[Y_{q_i}^2] = \frac{2K_n D}{h} \quad (10)$$

$$q_1 - E[Y_{q_1}] = q_2 - E[Y_{q_2}] = \dots = q_n - E[Y_{q_n}]. \quad (11)$$

Theorem 1 reveals an important property of the optimal order quantities of the random capacity EOQ model with multiple suppliers. If we reconsider the condition in (11), we see that  $q_i - E[Y_{q_i}]$  is a positive constant since  $Y_{q_i} \leq q_i$  for all  $q_i$ . This optimal policy stipulates that the retailer's order should be diversified such that the expected number of unfulfilled order units is the same for each supplier. This relation specifies the optimal order quantities from all suppliers as a function of the optimal

order quantity from any given supplier. Since  $q_i - E[Y_{q_i}]$  is an increasing function of  $q_i$ , if the order quantity from any supplier increases, then all other order quantities should also increase in order to satisfy (11). It is (10) which determines what should be the optimal order quantity for the first supplier and thereby establishes the optimal order quantities for the other suppliers as well. We see in (10) that as  $2K_n D/h$ , the square of classical EOQ formula, increases, all the order quantities increase.

If the setup cost is not dependent on the number of suppliers, it is always better to diversify among all available suppliers since setting any of the order quantities to zero would necessitate setting all order quantities to zero so that (10) is satisfied. This is quite intuitive, since by diversifying our orders we decrease the probability that we exceed the capacities of the suppliers and thereby decrease the uncertainty in what we receive.

In the next corollary we show that the optimal order quantities from the suppliers follow the stochastic order of their capacity distributions. The corollary states that if the probability of fully receiving the order is greater for a supplier compared to another, then we should order more from this supplier.

**Corollary 2.** *Let  $A_i \succeq_{st} A_j$  then  $q_i^* \geq q_j^*$ .*

**Proof.** By definition, if  $A_i \succeq_{st} A_j$ , then  $\bar{F}_i(u) \geq \bar{F}_j(u)$  for all  $u \geq 0$ . Since  $A_i \succeq_{st} A_j$  implies that  $\text{Min}(A_i, q) \succeq_{st} \text{Min}(A_j, q)$ , we can write  $E[\text{Min}(A_i, q)] \geq E[\text{Min}(A_j, q)]$  or simply  $E[Y_{q_i}]|_{q_i=q} \geq E[Y_{q_j}]|_{q_j=q}$ . Since  $q_k - E[Y_{q_k}]|_{q_k=q}$  is a non-decreasing function of  $q$  for any supplier  $k$ ,  $q_i^* \geq q_j^*$  so that the optimality condition (11) is satisfied. ■

We should remark that the underlying assumption behind the theory presented in this section is that the random capacities of the suppliers are unbounded. When bounded capacities are considered, the system of equations given in Theorem 1 may not yield a solution. This is due to the fact that even if the retailer orders more than the capacity bound from a supplier, the distribution of the quantity received will be the same as when the capacity bound is ordered. Thus,  $q_i - E[Y_{q_i}]$  will be constant after the bound is reached for supplier  $i$ . This means that it may be impossible to set the expected number of unfulfilled order units to the level satisfying (11). Then, the order quantity for that supplier should be set to its capacity bound, and the optimization problem should be solved for the rest of the supplier order quantities. Yet, one should note that Corollary 2 is always valid irrespective of the boundedness of the capacity distributions. The approach in the case of multiple suppliers with bounded capacities is illustrated in the next section.

### 3. SUPPLIERS WITH UNIFORM CAPACITIES

In real life, the suppliers have always a bound on their capacity which means that they can never satisfy an infinite order. Thus, it is practically relevant to consider

bounded capacity distributions. Among the bounded distributions, the uniform distribution is the simplest –yet reasonable– distribution to model the random capacity. The underlying assumption while using the uniform capacity distribution is that any capacity value within a given interval is equally likely. In the case of lack of knowledge about the true capacity distribution, the uniform distribution is the best choice from a practical point of view. Furthermore, noting that the supplier may be unable to send any quantity from time to time due to production shutdowns, etc., one can claim that the lower bound on the capacity distribution has to be zero.

When the random capacity of a supplier is uniformly distributed on  $[0, a]$  for some  $a > 0$ , by using (5) and (8) we can easily obtain

$$E[Y_q] = q - \frac{q^2}{2a} = q \left(1 - \frac{q}{2a}\right) \quad (12)$$

and

$$E[Y_q^2] = q^2 - \frac{2}{3a}q^3 = q^2 \left(1 - \frac{2q}{3a}\right). \quad (13)$$

In the single supplier case with  $n = 1$ , the optimality condition (10) yields

$$w(q) = q^2 \left(1 - \frac{q}{3a}\right) = \frac{2KD}{h}. \quad (14)$$

for  $0 \leq q \leq a$ . It is not surprising at all that  $\lim_{a \rightarrow +\infty} q^2 \left(1 - \frac{q}{3a}\right) = q^2$ , leading to the classical EOQ model. One can show that  $w(q)$  is strictly increasing on  $[0, a]$  with  $w(0) = 0$  and  $w(a) = 2a^2/3$ . Therefore, the optimal order quantity is the unique solution of (14) in  $[0, a]$  if  $a^2 \geq 3KD/h$ . However, if  $a^2 < 3KD/h$ , then the optimal order quantity is  $a$  since it does not make sense to order more than what the supplier can possibly deliver.

Note that our assumption  $\bar{F}(x) > 0$  is clearly not satisfied in the uniform capacity case since  $\bar{F}(x) = 0$  whenever  $x \geq a$ . Therefore, it is no longer true that the optimal order quantity is the unique solution of (14). However, this does not constitute a major obstacle. It suffices to treat this problem as a constrained optimization problem with  $0 \leq q \leq a$  so that whenever there is no solution of (14) on  $[0, a]$  the optimal order quantity is  $a$ .

We now consider the general case with  $n$  suppliers where the capacity  $A_i$  of supplier  $i$  is uniformly distributed between zero and some  $a_i > 0$ . In this section, we suggest a procedure to find the optimal order quantities when the random capacities have upper bounds.

**Theorem 3.** *Consider the EOQ model with  $n$  suppliers that have uniform capacities so that  $A_i \sim \text{Uniform}[0, a_i]$  for some  $a_i > 0$  and  $i = 1, 2, \dots, n$ . Let  $q_i^0$  be a solution*

of the equations

$$\left[ \frac{n(n-1)}{4a_1^2} \right] q_1^4 + \left[ \frac{2-3n}{3a_1} \sum_{i=1}^n \sqrt{\frac{a_i}{a_1}} \right] q_1^3 + \left[ 2 \sum_{i=1}^n \sum_{j=i}^n \sqrt{\frac{a_i a_j}{a_1 a_1}} - \sum_{i=1}^n \frac{a_i}{a_1} \right] q_1^2 = \frac{2K_n D}{h} \quad (15)$$

and

$$q_k = \sqrt{\frac{a_k}{a_1}} q_1 \quad (16)$$

for  $k = 2, 3, \dots, n$ . If  $q_i^0 \leq a_i$  for all  $i = 1, 2, \dots, n$ , then the optimal order quantity is  $Q_i = q_i^0$  for all  $i$ .

Theorem 3 states that, if all the order quantities given by the first order conditions do not exceed the capacity bounds, then the optimal order quantities are uniquely determined. Otherwise, the optimal order quantity for that supplier must be equal to its capacity bound since there is no point in ordering more. In Corollary 4, we state a related property of the optimal solution if the first order conditions do not lead to a feasible solution. It is interesting that the optimality condition is a cubic equation for  $n = 1$  and quartic equation for any larger  $n$  as stated in (14) and (15).

**Corollary 4.** Suppose without loss of generality that the suppliers are ordered such that  $a_1 \geq a_2 \geq \dots \geq a_n$ . If  $k = \max\{i; q_i^0 < a_i, i = 1, 2, \dots, n\}$ , then  $Q_i = a_i$  for all  $i = k + 1, k + 2, \dots, n$ .

**Proof.** It is sufficient to show that if  $q_i^0 > a_i$ , then  $q_{i+1}^0 > a_{i+1}$ . Using (16), we can write

$$q_{i+1}^0 = \sqrt{\frac{a_{i+1}}{a_1}} q_1^0 = \sqrt{\frac{a_{i+1}}{a_i}} \sqrt{\frac{a_i}{a_1}} q_1^0 = \sqrt{\frac{a_{i+1}}{a_i}} q_i^0 \quad (17)$$

Since  $a_i \geq a_{i+1}$ ,  $\sqrt{a_{i+1}/a_i} \geq a_{i+1}/a_i$ . Also noting that  $q_i^0 > a_i$ , (17) leads to

$$q_{i+1}^0 \geq \frac{a_{i+1}}{a_i} q_i^0 > \frac{a_{i+1}}{a_i} a_i = a_{i+1}. \quad (18)$$

Thus,  $q_{i+1}^0 > a_{i+1}$  and the rest follows by induction on  $i$ . ■

By Corollary 4, if the order quantities determined by the first order conditions in Theorem 3 are less than or equal to the capacity bounds for the first  $k$  high capacity suppliers, then the optimal order quantities for all the other suppliers are at their capacity bounds, i.e.,  $Q_i = a_i$  since  $q_i^0 \geq a_i$  for  $i = k + 1, \dots, n$ . So, by setting  $Q_{k+1} = a_{k+1}, \dots, Q_n = a_n$ , the first order conditions defined by (10) have to be resolved to find the optimal order quantities  $Q_1, Q_2, \dots, Q_k$ . This implies that we need to replace (15) by



$$\left[ \frac{k(k-1)}{4a_1^2} \right] q_1^4 + \left[ \frac{2-3k}{3a_1} \sum_{i=1}^k \sqrt{\frac{a_i}{a_1}} \right] q_1^3 + \left[ 2 \sum_{i=1}^k \sum_{j=i}^k \sqrt{\frac{a_i a_j}{a_1 a_1}} - \sum_{i=1}^k \frac{a_i}{a_1} \right] q_1^2 = \frac{2K_n D}{h}. \tag{19}$$

Once  $Q_1$  is found by using (19),  $Q_2, \dots, Q_k$  are determined by using the first order conditions in (16). Let us note that this will lead to an increase in the previously obtained order quantities  $q_1^0, q_2^0, \dots, q_k^0$  found by using Theorem (3) and this iterative procedure is repeated until  $Q_i \leq a_i$  for all  $i$ .

In Table 1 we provide some numerical illustrations for the two suppliers problem with uniform capacities. Experiments are made in two sets for  $a_1 = 85$  and  $a_1 = 100$  respectively and the corresponding  $a_2$  levels are chosen such that  $a_1 \geq a_2$ . The cost parameters are  $K = 200, D = 32, h = 2$ .

As a consequence of Corollary 4, if the higher capacity supplier is ordered at its capacity bound, (i.e.,  $Q_1 = a_1$ ), then so is the second supplier. It is also easy to check that the expected number of unfulfilled order units from both suppliers  $Q_i - E[Y_{Q_i}]$  are the same if both order quantities are lower than the capacity bounds. If the order quantity is at capacity bound, then  $E[Y_{Q_i}] = E[A_i] = a_i/2$ . Finally, if the capacity constraints are more restrictive due to low levels of  $a_2$ , then the optimal expected total cost is increased which is intuitive as well.

Table 1: Numerical Results of the Two Vendors Problem with Uniform Capacities

$a_1$	$a_2$	$Q_1$	$Q_2$	$Q_1 - E[Y_{Q_1}]$	$Q_2 - E[Y_{Q_2}]$	$E[Y_{Q_1}]$	$E[Y_{Q_2}]$	$TC$
85	85	49.44	49.44	14.38	14.38	35.06	35.06	169.01
	50	61.86	47.45	22.51	22.51	39.35	24.93	173.19
	30	75.26	30.00	32.32	15.00	41.94	15.00	180.52
	25	85.00	25.00	42.50	12.50	42.50	12.50	183.26
	10	85.00	10.00	42.50	5.00	42.50	5.00	195.09
	5	85.00	5.00	42.50	2.50	42.50	2.50	200.65
	2	85.00	2.00	42.50	1.00	42.50	1.00	204.48
100	100	47.40	47.40	11.23	11.23	36.17	36.17	167.14
	50	61.21	43.28	18.73	18.73	42.48	24.55	171.51
	30	73.23	30.00	26.81	15.00	46.42	15.00	176.45
	20	80.34	20.00	32.28	10.00	48.07	10.00	180.69
	10	100.00	10.00	50.00	5.00	50.00	5.00	186.67
	5	100.00	5.00	50.00	2.50	50.00	2.50	190.32
	2	100.00	2.00	50.00	1.00	50.00	1.00	192.84

These illustrations help us provide some intuition for the multiple suppliers problem with bounded capacities: If the optimal order quantities for certain suppliers are at their bounds, then there is a capacity problem at these suppliers which prevents

us to diversify our orders as in the absence of bounds. Obviously, this leads to an increase in the average costs. We can overcome this difficulty by increasing the number of suppliers we are working with.

#### 4. SUPPLIERS WITH EXPONENTIAL CAPACITIES

In this section, we assume that the random capacities are exponentially distributed with parameter  $\mu$  so that  $F(x) = 1 - \exp(-\mu x)$ ,  $x \geq 0$ . Exponential capacities provide us an ideal setting for getting insight about unbounded capacity case since it is computationally more tractable compared to other unbounded distributions. Specifically, exponential case yields some explicit solutions. The analysis of the exponential capacities provided in this section is then used for an illustration of unbounded capacities case in contrast with the bounded capacities case, discussed in the previous section.

By using (5) and (8) one can easily show that

$$E[Y_q] = \frac{1}{\mu}(1 - e^{-\mu q}) \quad (20)$$

and

$$E[Y_q^2] = \frac{2}{\mu^2}(1 - (1 + \mu q)e^{-\mu q}). \quad (21)$$

In the single supplier case with  $n = 1$ , the optimal order quantity satisfies (10) as

$$\frac{2(\mu Q + e^{-\mu Q} - 1)}{\mu^2} = \frac{2KD}{h}. \quad (22)$$

It is not surprising at all that  $\lim_{\mu \rightarrow 0} \frac{2(\mu Q + e^{-\mu Q} - 1)}{\mu^2} = Q^2$ , leading to the classical EOQ model. As a matter of fact, it is not difficult to show that, as  $\mu$  gets smaller, the random capacity gets stochastically larger and the order quantity gets smaller.

The solution of (22) is given explicitly by

$$Q = \frac{1}{\mu} \left[ 1 + \hat{K}\mu^2 + W(-e^{-(1+\hat{K}\mu^2)}) \right] \quad (23)$$

where  $\hat{K} = KD/h$  for notational simplicity and  $W$  is the Lambert  $W$  function that can be computed with arbitrary precision. The Lambert  $W$  function is the inverse of  $f(y) = ye^y$  so that it satisfies  $W(x)e^{W(x)} = x$  for any  $x$ . It arises naturally in some interesting problems and has some nice properties. The reader is referred to Corless et al. [1996] for details on Lambert  $W$  function. In our case, since  $0 \leq \hat{K} \leq +\infty$ , we have  $-e^{-1} \leq -e^{-(1+\hat{K}\mu^2)} \leq 0$  and

$$-1 = W(-e^{-1}) \leq W(-e^{-(1+\hat{K}\mu^2)}) \leq W(0) = 0 \quad (24)$$

so that

$$\hat{K}\mu \leq Q \leq \frac{1}{\mu} + \hat{K}\mu. \quad (25)$$

Using (22) and (25) for small and large values of  $\mu$ , we can use the approximation

$$Q \cong \begin{cases} KD\mu/h, & \text{for large } \mu \\ \sqrt{2KD/h} & \text{for small } \mu \end{cases} \quad (26)$$

without having to use the Lambert  $W$  function. Using the fact that  $Q \geq EOQ$  (Wang and Gerchak [1996b]) and (25), we can obtain bounds  $L$  and  $U$  for  $Q$  such that

$$L = \max \left\{ \sqrt{\frac{2KD}{h}}, \frac{KD\mu}{h} \right\} \leq Q \leq \frac{1}{\mu} + \frac{KD\mu}{h} = U. \quad (27)$$

We can rewrite the bounds as

$$[L, U] = \left[ \sqrt{\frac{2KD}{h}} \max \left\{ 1, \sqrt{\frac{\hat{K}}{2}} \mu \right\}, \frac{1}{\mu} + \hat{K}\mu \right] \quad (28)$$

so that the lower bound is given by the EOQ if  $\mu \leq \sqrt{2/\hat{K}}$ .

As a numerical illustration, suppose that there is a single supplier with an exponentially distributed random capacity. The numerical results are given in Table 2 for a selected set of values. The table also includes previously developed bounds  $[L, U]$  where  $L = \max\{\sqrt{2KD/h}, KD\mu/h\}$ ,  $U = (1/\mu) + KD\mu/h$  and the EOQ ( $\sqrt{2KD/h}$ ).

Table 2: Numerical Results of the Single Vendor Problem with Exponential Capacity

No	$K$	$D$	$h$	$\mu$	$Q$	$TC(Q)$	$100E[T]$	$[L, U]$	EOQ
1	1	100	3	0.025	8.5	25.4	7.6	[8.2, 40.8]	8.2
2	50	100	3	0.025	75.6	226.9	33.9	[57.7, 81.7]	57.7
3	100	100	3	0.025	121.4	364.2	38.1	[83.3, 123.3]	81.7
4	50	10	1	0.025	36.4	36.4	238.9	[31.6, 52.5]	31.6
5	50	20	1	0.025	54.9	54.9	149.2	[44.7, 65]	44.7
6	50	500	1	0.025	665.0	665.0	8.0	[625, 665]	223.6
7	50	100	0.1	0.025	1290.0	129.0	40.0	[1250, 1290]	316.2
8	50	100	1	0.025	164.3	164.3	39.3	[125, 165]	100
9	50	100	10	0.025	36.4	363.9	23.9	[31.6, 52.5]	31.6
10	1	100	3	0.25	12.1	36.4	3.8	[8.3, 12.3]	8.2
11	1	100	3	0.50	18.7	56.0	2.0	[16.7, 18.7]	8.2
12	1	100	3	1.5	50.7	152.0	0.7	[50, 50.7]	8.2

One can observe the sensitivity of the optimal solution on the model parameters. The optimal order quantity  $Q$  increases as the order cost  $K$  or demand  $D$  or parameter  $\mu$  increases, or the holding cost  $h$  decreases. Note that the cycle length  $E[T]$  does not necessarily increase with  $Q$  as in the deterministic EOQ model. Due to the

randomness of the capacity, it also depends on how  $\mu$  or  $D$  varies in addition to the change in  $Q$ . The approximations provided by (26) are quite good since the interval  $[L, U]$  is short for large  $\mu$  as in case 12. For small values of  $\mu$ , the EOQ ( $\sqrt{2KD/h}$ ) may provide a good approximation as in case 1 depending on the value of  $KD/h$ . The approximation will be better for smaller values of this ratio.

Now let us consider the general case where there are multiple suppliers with exponentially distributed capacities and derive some properties of the optimal solution. Before we state Theorem 5 consider

$$f(q) = q - \frac{1}{\mu_1}(1 - e^{-\mu_1 q}) \quad (29)$$

as the difference between  $q$  and  $E[Y_q]$  for the first supplier.

**Theorem 5.** *Consider the EOQ model with  $n$  suppliers that have exponential capacities so that  $A_i \sim \text{Exponential}(\mu_i)$  for some  $\mu_i > 0$  and  $i = 1, 2, \dots, n$ . The optimal order quantities can be found as the unique nonnegative solution of the equations*

$$\begin{aligned} \frac{2K_n D}{h} &= \frac{2}{\mu_1^2} (\mu_1 q_1 + e^{-\mu_1 q_1} - 1) + \sum_{i=2}^n \frac{2}{\mu_1 \mu_i} \left( \mu_1 q_1 + (1 - e^{-\mu_1 q_1}) W \left( -e^{-1 - \mu_i f(q_1)} \right) \right) \\ &+ 2 \sum_{i=2}^{n-1} \sum_{j=i+1}^n \frac{1}{\mu_i} \frac{1}{\mu_j} \left( 1 + W \left( -e^{-1 - \mu_i f(q_1)} \right) \right) \left( 1 + W \left( -e^{-1 - \mu_j f(q_1)} \right) \right) \quad (30) \end{aligned}$$

and

$$q_k = f(q_1) + \frac{1}{\mu_k} \left[ 1 + W \left( -e^{-(1 + \mu_k f(q_1))} \right) \right] \quad (31)$$

for  $k = 2, 3, \dots, n$ .

Note that when there is a single supplier, (30) is equivalent to (22). Theorem 5 gives a simple procedure to compute the optimal order quantities. First, the optimal order quantity is determined for the first supplier as the unique nonnegative solution of (30); then (31) is solved to compute the optimal order quantities for all other suppliers.

We provide a numerical illustration in Table 3 for 2 suppliers with the same data used for the uniform suppliers case where the results are given in Table 1. Recall that the cost parameters are  $K_2 = 200$ ,  $D = 32$  and  $h = 2$  with EOQ = 80. We choose the parameters of the exponential distribution so that means are the same when compared with the uniform case. In other words, we set  $1/\mu_i = a_i/2$  so that  $2/\mu_i = a_i$  and one can compare the results in the two tables to see the effects of the uniform and exponential distributions. Since there are no upper bounds on the order quantities as in the uniform case, the optimal solution is determined uniquely by (30) and (31). Note also that  $Q_i - E[Y_{Q_i}]$  is always the same for both suppliers unlike the uniform case. As the mean capacity of the supplier 2 is decreased by increasing  $\mu_2$ ,

the expected quantity  $E[Y_{Q_2}]$  that is actually received also decreases steadily. But, note that the order quantity eventually gets close to the mean, which was also the case for the uniform distribution.

Table 3: Numerical Results of the Two Vendors Problem with Exponential Capacities

$2/\mu_1$	$2/\mu_2$	$Q_1$	$Q_2$	$Q_i - E[Y_{Q_i}]$	$E[Y_{Q_1}]$	$E[Y_{Q_2}]$	$TC$
85	85	57.48	57.48	25.97	31.51	31.51	177.97
	50	70.27	58.50	35.91	34.37	22.59	185.73
	30	82.74	61.05	46.31	36.48	14.74	194.98
	25	86.82	62.25	49.83	36.99	12.41	198.47
	10	101.94	68.30	63.30	38.64	5.00	213.88
	5	108.10	71.44	68.94	39.16	2.50	221.19
	2	112.12	73.66	72.66	39.46	1.00	226.23
100	100	54.13	54.13	21.07	33.07	33.07	174.39
	50	69.08	53.73	31.64	37.44	22.09	182.34
	30	80.20	54.86	40.25	39.24	14.61	189.62
	20	87.83	56.43	46.46	41.37	9.96	195.59
	10	97.12	59.29	54.29	42.83	5.00	204.24
	5	102.43	61.38	58.88	43.55	2.50	209.87
	2	105.86	62.88	61.88	43.98	1.00	213.73

## 5. DIVERSIFICATION UNDER IDENTICAL SUPPLIERS

We now discuss the case when all supplier capacities are independent and identically distributed. The results for the general case obviously apply when the capacities are identical. Yet one can not clearly identify the effect of the number of suppliers on the optimal diversification scheme for nonidentical suppliers, since the scheme is also dependent on the individual yield structures of different suppliers. By considering identical suppliers case, one can isolate the effect of the number of suppliers on the optimal order policy.

Since the suppliers are identical  $E[Y_{q_i}] = E[Y_{q_j}]$  and  $E[Y_{q_i}^2] = E[Y_{q_j}^2]$  for all  $i, j$ , the symmetric nature of the formulation reduces our multivariate optimization problem in (4) to one involving the single decision variable  $q = q_1 = q_2 = \dots = q_n$  with average total cost

$$\min_{q \geq 0} TC(q) = \frac{K_n D}{nE[Y_q]} + \frac{1}{2}h \left[ \frac{E[Y_q^2] + (n-1)E[Y_q]^2}{E[Y_q]} \right]. \quad (32)$$

**Theorem 6.** *The optimal common order quantity  $Q_n$  of the random capacity EOQ model with  $n$  identical suppliers is the unique finite solution of the following equation*

$$2nqE[Y_q] + n(n-1)E[Y_q]^2 - nE[Y_q^2] = \frac{2K_n D}{h}. \quad (33)$$

Now we derive and illustrate the effects of diversification among the suppliers when they are identical with uniformly and exponentially distributed capacities respectively. Note that the total order quantity is  $nQ_n$  and it is an important quantity to measure the effects of diversification. We assume that  $K_n = K$  so that the cost of ordering is a constant irrespective of the number of suppliers. However we also illustrate the case where  $K_n = K + nk$ ,  $K, k \geq 0$ .

**5.1. Suppliers with Uniform Capacities.** When  $a_k = a$ , then  $q_k = q$  for all  $k$  and the optimality condition for the identical uniform capacities problem (33) can be rewritten as

$$w_n(q) = \left[ \frac{n(n-1)}{4a^2} \right] q^4 + \left[ \frac{(2-3n)n}{3a} \right] q^3 + n^2 q^2 = \frac{2KD}{h} \quad (34)$$

using (12)-(14). One can show that  $w_n(q)$  is strictly increasing on  $[0, a]$  with  $w_n(0) = 0$  and  $w_n(a) = ((3n^2 + 8n - 3)/12)a^2$ . Therefore, the optimal order quantity is the unique nonnegative solution of (34) in  $[0, a]$  if  $a^2 \geq 24KD/(3n^2 + 8n - 3)h$ . However, if  $a^2 < 24KD/(3n^2 + 8n - 3)h$ , then the optimal order quantity is  $a$ . Let

$$n^* = \min\{n \geq 1 : 3n^2 + 8n - 3 \geq \frac{24KD}{a^2h}\} \quad (35)$$

then it follows that  $Q_n = a$  and the total order quantity is  $nQ_n = na$  whenever  $n < n^*$ . This implies that the total order quantity increases on  $[1, n^*]$  because the suppliers do not have sufficient total capacity and one should order the maximum amount possible. However, once the number of suppliers  $n$  exceeds  $n^*$ , the total order quantity decreases.

**Corollary 7.** *Suppose there are  $n$  identical suppliers with uniformly distributed capacities so that  $A_i \sim U[0, a]$  for some  $a > 0$  and all  $i = 1, 2, \dots, n$ . If  $K_n = K$  for all  $n \geq 1$ , then*

$$(n+1)Q_{n+1} \leq nQ_n. \quad (36)$$

for all  $n \geq n^*$  and the total order quantity decreases as  $n$  increases. Moreover,  $\lim_{n \rightarrow +\infty} nQ_n = \text{EOQ} = \sqrt{2KD/h}$  for any  $a$ .

We now provide some illustrations in Figure 1, where a replenishment order is given simultaneously to  $n$  identical suppliers, each having uniformly distributed capacities on  $[0, a]$ . We see that as the number of identical suppliers increases, the total optimal order quantity  $nQ_n$  decreases and converges to the optimal order quantity of the incapacitated or certain yield model with  $\text{EOQ} = \sqrt{2KD/h} = 80$ . In other words, if the number of suppliers is large, then there is almost no risk of getting stockout at any supplier, thus all orders are fully received and therefore it is optimal to diversify the  $\text{EOQ} = 80$  between  $n$  suppliers.

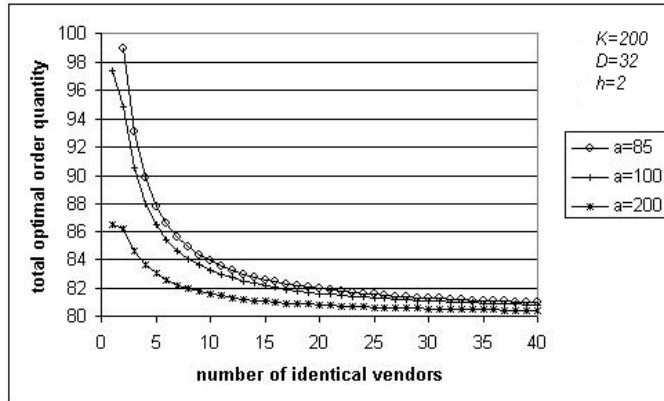


Figure 1: Total Optimal Order Quantity versus Number of Identical Suppliers

It is obvious that diversification decreases the total order quantity more significantly when the maximum capacity is closer to the  $EOQ = 80$ . If all suppliers have infinite capacities. Then, the yield is certain and optimal policy is to order  $EOQ = 80$  from a single supplier. However, if the capacity is random and bounded by  $a = 200$ , then the order quantity is  $Q_1 = 86$  which is greater than the  $EOQ$ . If the capacity bound is even less than 200, say  $a = 100$ , then the optimal order quantity increases to  $Q_1 = 97$ .

In Figure 2, the effect of the diversification on the optimal expected cost is illustrated. As the number of identical suppliers increases, the optimal expected cost decreases and converges to the cost of an infinite capacity  $EOQ$  model given by  $\sqrt{2KDh} = 160$ . In our model, diversification with a large number of suppliers is always beneficial, allowing order quantities and expected costs to decrease since the setup cost  $K = 200$  is not related to the number of suppliers used.

Figure 3 depicts the behavior of the expected total yield  $nE[Y_{Q_n}]$  as a function of the number of identical suppliers used. It is interesting to observe that as  $n$  increases, the expected total yield  $nE[Y_{Q_n}]$  increases too, although the total order quantity  $nQ_n$  decreases. Though no formal proof is provided, we can see that it is due to the reduced probability of getting stockout at any supplier, when less is ordered from each. It follows by observing the 3–standard deviation bounds of the total random yield that when more suppliers are used, risk is diversified and this leads to a decrease in the variability of the total yield. Figures 1 and 3 show that when the orders are diversified, less is ordered but more is expected and they both converge to the  $EOQ$  with less deviation when  $n$  is very large.

**5.2. Suppliers with Exponential Capacities.** Following the similar discussion for the uniform capacities case, now we consider identical suppliers with exponential

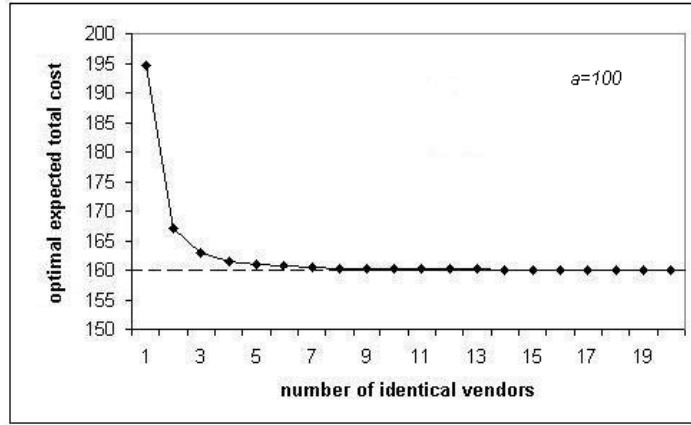


Figure 2: Optimal Expected Total Cost versus Number of Identical Suppliers

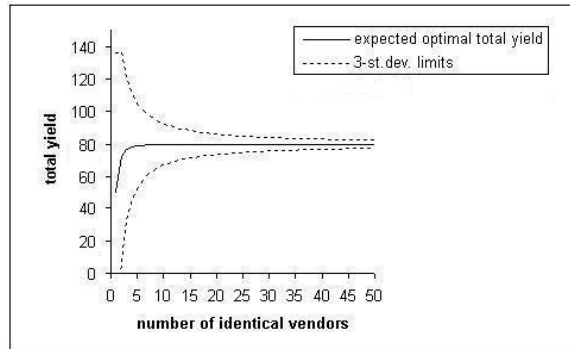


Figure 3: Total Expected Yield versus Number of Identical Suppliers



capacities and illustrate the effect of diversification.

If there are two identical suppliers, then by using (33) with  $n = 2$  we obtain

$$\frac{2\mu Q_2 + e^{-2\mu Q_2} - 1}{\mu^2} = \frac{K_2 D}{h} \quad (37)$$

where  $Q_2$  is the optimal order quantity for each one of the two identical suppliers. Now, the solution is

$$Q_2 = \frac{1}{2\mu} \left[ 1 + \hat{K}_2 \mu^2 + W(-e^{-(1+\hat{K}_2 \mu^2)}) \right] \quad (38)$$

where  $\hat{K}_2 = K_2 D/h$ .

The comparison of (23) and (38) indicates that if  $K_2 = K_1 = K$  so that  $\hat{K}_2 = \hat{K}_1$ , then  $2Q_2 = Q_1$ . This is an amazing result which states that the total order quantity is the same for the single and double supplier models. The ordering cost is the same in both cases and the order is diversified between the two suppliers by distributing the order quantity equally among them. However, if  $K_2 \geq K_1$ , then one can easily verify that  $2Q_2 \geq Q_1$  by using the fact that  $W$  is increasing on  $[-e^{-1}, 0]$ .

For arbitrary  $n \geq 1$ , the optimality condition (33) yields the unique solution of  $Q_n$  as

$$\frac{\mu n Q_n - n(n-2)e^{-\mu Q_n} + 0.5n(n-1)e^{-2\mu Q_n} + 0.5n(n-3)}{\mu^2} = \frac{K_n D}{h}. \quad (39)$$

**Corollary 8.** *Suppose there are  $n$  identical suppliers having exponentially distributed capacities with parameter  $\mu > 0$ . If  $K_n = K$  for all  $n \geq 1$ , then*

$$(n+1)Q_{n+1} \leq nQ_n \quad (40)$$

so that the total order quantity decreases as  $n$  increases. Moreover,  $\lim_{n \rightarrow +\infty} nQ_n = EOQ = \sqrt{2KD/h}$  for any  $\mu$ .

Corollary 8 implies that, if the fixed ordering cost does not increase, then the total order quantity decreases with the number of suppliers. Apparently, diversification of suppliers decreases the risk of having insufficient stock at a single supplier.

**5.3. How Many Suppliers?.** A very interesting and relevant issue in a retailer's diversification among suppliers is the selection of the suppliers to work with. One would need a model that allows different unit prices for different suppliers in order to discriminate among suppliers. But one can still address the issue of how many suppliers should a retailer work with within the confines of our model. In this section we assume that all suppliers are identical so that we can isolate the effect of number of suppliers used.

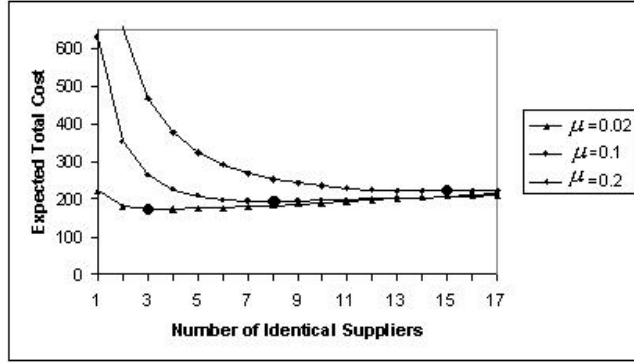


Figure 4: Expected Total Cost versus Number of Identical Suppliers

We assume that the ordering cost,  $K_n$ , changes with the number of suppliers,  $n$  in the form of  $K + kn$  where  $K$  is the fixed ordering cost and there is a variable cost of ordering,  $k$ ,  $k > 0$  for each supplier used. We have already shown that the total cost decreases with the number of suppliers used when the ordering cost is independent of the number of suppliers used. Thus, it is optimal to use all available suppliers. However, we now show that if the ordering cost increases linearly with the number of suppliers, there is an optimal number of suppliers which may be less than the number of available suppliers .

We consider  $n$  identical suppliers having exponentially distributed capacities with equal parameter  $\mu$ . Letting  $K_n = K + kn$  in (39), the optimal order quantity,  $q_n$  is found and the expected total cost is plotted for different levels of  $n$  and  $\mu$  where  $D = 32$ ,  $K = 180$ ,  $k = 10$ ,  $h = 2$ . It is observed from Figure 4 that the benefit of increasing the number of suppliers is large initially, but eventually decreases and then turns negative, in agreement with intuition. This implies that there exists an optimal number of suppliers. It is clear that when the capacity parameter,  $\mu$  increases (i.e. the expected capacity,  $1/\mu$  decreases), the optimal number of suppliers to work with increases in order to reduce the stockout risk at any supplier. Moreover, the optimal expected total cost is greater when the expected capacities are lower (i.e.,  $\mu$  is greater).

## 6. CONCLUSION

In this study, our objective is to introduce an EOQ model with multiple suppliers and random capacities that enables us to obtain intuition for ordering policy decisions. It is remarkable to see that when the optimal policy is applied, the expected number of unfulfilled order units from all suppliers must all be the same. Noting that this property applies only for the unbounded random capacity problem, the model with bounded capacities is analyzed under the special case of uniformly distributed capac-

ities. If the solution of the first order condition violates the capacity constraint for any supplier, then it is optimal to order at the capacity for that supplier and for all suppliers with lower capacity bounds. Moreover, as an illustration of the unbounded capacity case, the exponential capacities problem is analyzed and a characterization for the optimal order quantities is obtained. Then the results for the two cases are compared and contrasted. Finally the effect of diversification is discussed under the settings of identical suppliers with uniformly distributed capacities and exponentially distributed capacities to show that the total optimal order quantity decreases and converges to *EOQ* as the number of suppliers increase. Furthermore when the ordering cost increases linearly with the number of suppliers, there is an optimal level for the number of suppliers used.

Our results indicate that computing the optimal order quantities is not very difficult especially when compared with the complexity of the problem. Given the stochastic structure of the random capacity, one can easily compute this quantity to almost arbitrary precision. However, we also like to point out that much needs to be done in improving the model. The obvious generalization would be to have suppliers with different ordering and holding costs. This presents a challenge in modelling as well as in the optimization. Characterizations on the optimal order quantities under this setting would be very helpful to the practitioner, especially if they present further intuition into the decision making process about how many and which suppliers to use. Finally, computational difficulties when the cost function is more complex will need to be resolved.

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### Appendix

**Proof. (Theorem 1):** We check the first order optimality conditions to find the optimal order quantity that minimizes  $TC(q_1, q_2, \dots, q_n)$ . Differentiating (4) with respect to  $q_k$  gives

$$\frac{\partial TC(q_1, q_2, \dots, q_n)}{\partial q_k} = \frac{\bar{F}_k(q_k)}{2\left(\sum_{i=1}^n E[Y_{q_i}]\right)^2} \left[ -2K_n D + h \left( 2(q_k + \sum_{i \neq k} E[Y_{q_i}]) \sum_{i=1}^n E[Y_{q_i}] - \sum_{i=1}^n E[Y_{q_i}^2] - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E[Y_{q_i}] E[Y_{q_j}] \right) \right] \quad (41)$$

Note that the denominator is always nonnegative and since  $\bar{F}_k(q_k) > 0$  for all  $q_k \geq 0$ , the second term of the numerator should be zero at the optimal solution so that

$$2(q_k + \sum_{i \neq k} E[Y_{q_i}]) \sum_{i=1}^n E[Y_{q_i}] - \sum_{i=1}^n E[Y_{q_i}^2] - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E[Y_{q_i}] E[Y_{q_j}] = \frac{2K_n D}{h}. \quad (42)$$

This gives us (10) by taking  $k = 1$  and cancelling the common terms. Note that we have  $n$  equations in the form of (42) where the right hand side is the constant  $2K_n D/h$ , which is in fact equal to  $\text{EOQ}^2$ . Thus, by equating these equations and cancelling the common terms, (11) follows directly.

We now show that the solution  $\{q_i^0 > 0; i = 1, 2, \dots, n\}$  satisfying the first order conditions defined by (10) and (11) is unique. By (11) we can see that for any fixed quantity  $q_1$ , the other order quantities are uniquely determined since  $q_i - E[Y_{q_i}]$  is zero when  $q_i = 0$  and it is an increasing function of  $q_i$ . Yet one can still freely choose any  $q_1$  and get other order quantities such that (11) is still satisfied. Thus we need to show that only one of these infinitely many solutions also satisfies (10). Using (6), the implicit differentiation of (11) yields

$$dq_1 - \bar{F}_1(q_1)dq_1 = dq_2 - \bar{F}_2(q_2)dq_2 = \dots = dq_n - \bar{F}_n(q_n)dq_n. \quad (43)$$

Since  $1 - \bar{F}_i(q_i) = F_i(q_i)$ , we have the relation

$$\frac{dq_i}{dq_1} = \frac{F_1(q_1)}{F_i(q_i)} \quad (44)$$

for all  $i, k$ . Note that by using (11), one can express all optimal order quantities as a function of  $q_1$ . So, we define the left hand side of (10) as  $w(q_1, q_2(q_1), \dots, q_n(q_1))$ . Obviously  $w(0, 0, \dots, 0) = 0$ , so if we can show that  $w$  is increasing for  $q_1 > 0$ , then the solution satisfying (10) is unique since the right hand side is a positive constant given by  $\text{EOQ}^2$ .

Using (6) and (9), the derivative  $dw(q_1, q_2(q_1), \dots, q_n(q_1))/dq_1$  can be obtained as

$$\begin{aligned} & 2 \left[ 1 + \sum_{i \neq 1} \bar{F}_i(q_i) \frac{dq_i}{dq_1} \right] \sum_{i=1}^n E[Y_{q_i}] + 2 \left( q_1 + \sum_{i \neq 1} E[Y_{q_i}] \right) \left( \sum_{i=1}^n \bar{F}_i(q_i) \frac{dq_i}{dq_1} \right) \\ & - \sum_{i=1}^n 2q_i \bar{F}_i(q_i) \frac{dq_i}{dq_1} - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \bar{F}_i(q_i) \frac{dq_i}{dq_1} E[Y_{q_j}] + E[Y_{q_i}] \bar{F}_j(q_j) \frac{dq_j}{dq_1}. \end{aligned} \quad (45)$$

After substituting (44) in (45), (45) becomes

$$\begin{aligned} & 2 \left[ 1 + \sum_{i \neq 1} \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) \right] \sum_{i=1}^n E[Y_{q_i}] + 2 \left[ q_1 + \sum_{i \neq 1} E[Y_{q_i}] \right] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) \\ & - \sum_{i=1}^n 2q_i \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[ \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) E[Y_{q_j}] + \frac{\bar{F}_j(q_j)}{F_j(q_j)} F_1(q_1) E[Y_{q_i}] \right]. \end{aligned} \quad (46)$$

Notice that the last term in (46) can be written as

$$\begin{aligned}
& \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[ \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) E[Y_{q_j}] + \frac{\bar{F}_j(q_j)}{F_j(q_j)} F_1(q_1) E[Y_{q_i}] \right] \\
&= \sum_{i=1}^n \sum_{j=1, j \neq i}^n \left( \frac{\bar{F}_i(q_i)}{F_i(q_i)} F_1(q_1) E[Y_{q_j}] \right). \\
&= F_1(q_1) \left[ \sum_{i=1}^n \sum_{j=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} E[Y_{q_j}] - \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} E[Y_{q_i}] \right] \\
&= F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \sum_{j=1}^n E[Y_{q_j}] - F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} E[Y_{q_i}].
\end{aligned} \tag{47}$$

By substituting (47) in (46) and by making further simplifications, we get

$$\begin{aligned}
& 2 \sum_{i=1}^n E[Y_{q_i}] + 2F_1(q_1) \sum_{i=1}^n E[Y_{q_i}] \sum_{i \neq 1} \frac{\bar{F}_i(q_i)}{F_i(q_i)} + 2q_1 F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \\
&+ 2F_1(q_1) \sum_{i \neq 1} E[Y_{q_i}] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} - 2F_1(q_1) \sum_{i=1}^n q_i \frac{\bar{F}_i(q_i)}{F_i(q_i)} \\
&- 2F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \sum_{j=1}^n E[Y_{q_j}] + 2F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} E[Y_{q_i}]
\end{aligned} \tag{48}$$

which can be written as

$$\begin{aligned}
& 2 \sum_{i=1}^n E[Y_{q_i}] - 2\bar{F}_1(q_1) \sum_{i=1}^n E[Y_{q_i}] + 2q_1 F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \\
&+ 2F_1(q_1) \sum_{i \neq 1} E[Y_{q_i}] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} - 2F_1(q_1) \sum_{i=1}^n q_i \frac{\bar{F}_i(q_i)}{F_i(q_i)} + 2F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} E[Y_{q_i}].
\end{aligned} \tag{49}$$

Rewriting the fourth term in (49) and reorganizing the terms gives

$$\begin{aligned}
& 2F_1(q_1) \sum_{i=1}^n E[Y_{q_i}] + 2q_1 F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \\
&+ 2F_1(q_1) \left( \sum_{i=1}^n E[Y_{q_i}] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} - E[Y_{q_1}] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \right) - 2F_1(q_1) \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} (q_i - E[q_i])
\end{aligned} \tag{50}$$

which is equal to

$$2F_1(q_1) \left\{ \sum_{i=1}^n E[Y_{q_i}] + \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} [(q_1 - E[Y_{q_1}]) - (q_i - E[q_i])] + \sum_{i=1}^n E[Y_{q_i}] \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \right\}. \tag{51}$$

Since  $(q_1 - E[Y_{q_1}]) = (q_i - E[q_i])$  by the first order conditions defined in (11), (51) is finally simplified as

$$2F_1(q_1) \sum_{i=1}^n E[Y_{q_i}] \left( 1 + \sum_{i=1}^n \frac{\bar{F}_i(q_i)}{F_i(q_i)} \right) \tag{52}$$

which is definitely positive for  $q_1 > 0$ . So,  $w(q_1, q_2(q_1), \dots, q_n(q_1))$  is an increasing function of  $q_1$ , and since  $w(0, 0, \dots, 0) = 0$ , there is a unique solution in the form  $[q_1^0, q_2^0, \dots, q_n^0]$  satisfying the first order conditions defined by (10) and (11).

In order to determine the nature of the critical point, we need to check the Hessian of  $TC(q_1, q_2, \dots, q_n)$  at this point. The second degree partial derivatives of the function at the critical point have to be computed. Taking the derivative of (41) with respect to  $q_k$  and using the fact that (41) is equal to zero at the critical point, we obtain

$$\begin{aligned} & \left. \frac{\partial^2 TC(q_1, q_2, \dots, q_n)}{\partial q_k^2} \right|_{(q_1^0, q_2^0, \dots, q_n^0)} \\ &= \frac{h\bar{F}_k(q_k^0)}{2\left(\sum_{i=1}^n E[Y_{q_i^0}]\right)^2} \cdot \left[ \begin{array}{c} 2\sum_{i=1}^n E[Y_{q_i^0}] + 2(q_k^0 + \sum_{i \neq k} E[Y_{q_i^0}])\bar{F}_k(q_k^0) - 2q_k^0\bar{F}_k(q_k^0) \\ -2\bar{F}_k(q_k^0)\sum_{i \neq k} E[Y_{q_i^0}] \end{array} \right] \\ &= \frac{h\bar{F}_k(q_k^0)}{\sum_{i=1}^n E[Y_{q_i^0}]} > 0. \end{aligned} \tag{53}$$

Taking the derivative of (41) with respect to  $q_j$  and, again, using the fact that (41) is equal to zero at the critical point, we obtain

$$\begin{aligned} & \left. \frac{\partial^2 TC(q_1, q_2, \dots, q_n)}{\partial q_k \partial q_j} \right|_{(q_1^0, q_2^0, \dots, q_n^0)} \\ &= \frac{h\bar{F}_k(q_k^0)}{2\left(\sum_{i=1}^n E[Y_{q_i^0}]\right)^2} \cdot \left[ \begin{array}{c} 2\bar{F}_j(q_j^0)\sum_{i=1}^n E[Y_{q_i^0}] + 2(q_k^0 + \sum_{i \neq k} E[Y_{q_i^0}])\bar{F}_j(q_j^0) - 2q_j^0\bar{F}_j(q_j^0) \\ -2\bar{F}_j(q_j^0)\sum_{i \neq j} E[Y_{q_i^0}] \end{array} \right] \\ &= \frac{h(\bar{F}_k(q_k^0))^2}{2\left(\sum_{i=1}^n E[Y_{q_i^0}]\right)^2} \left[ (q_k^0 - E[Y_{q_k^0}]) - (q_j^0 - E[Y_{q_j^0}]) \right]. \end{aligned} \tag{54}$$

Since  $(q_k - E[Y_{q_k}]) = (q_j - E[Y_{q_j}])$  at the critical point by the first order conditions given in (11), (54) is equal to zero. Hence, all the off-diagonal entries of the Hessian are 0, while all the diagonal entries are positive. This means that the Hessian is a positive definite matrix and the unique critical point defined by (10) and (11) is a local minimum. Moreover, the partial derivatives of the total cost function indicate that at any boundary (i.e., where  $q_i = 0$ ), the total cost function decreases as one gets away from the boundary (i.e., as  $q_i$  increases). Thereby, the minimum has to occur inside the feasible region. Thus, the critical point is the global minimum for the  $TC(q_1, q_2, \dots, q_n)$  function. ■

**Proof. (Theorem 3):** Since  $A_i \sim \text{Uniform}[0, a_i]$ ,  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  can be written as in (12) and (13). Now, consider the first order conditions given in Theorem 1. By using (12) and (13), (11) gives the first order optimality condition in (16). By

further substitution, we rewrite  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  as

$$E[Y_{q_i}] = \sqrt{\frac{a_i}{a_1}} q_1 - \frac{q_1^2}{2a_1} \quad (55)$$

$$E[Y_{q_i}^2] = \frac{a_i}{a_1} q_1^2 - \frac{2}{3a_1} \sqrt{\frac{a_i}{a_1}} q_1^3. \quad (56)$$

Now, we substitute  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  in (10) to derive the first order optimality condition in (15). This leads to

$$\begin{aligned} \frac{2K_n D}{h} &= 2q_1 \sum_{i=1}^n \left( \sqrt{\frac{a_i}{a_1}} q_1 - \frac{q_1^2}{2a_1} \right) + 2 \sum_{i=2}^n \sum_{j=i}^n \left( \sqrt{\frac{a_i}{a_1}} q_1 - \frac{q_1^2}{2a_1} \right) \left( \sqrt{\frac{a_j}{a_1}} q_1 - \frac{q_1^2}{2a_1} \right) \\ &\quad - \sum_{i=1}^n \left( \frac{a_i}{a_1} q_1^2 - \frac{2}{3a_1} \sqrt{\frac{a_i}{a_1}} q_1^3 \right) \\ &= \left[ \frac{n(n-1)}{4a_1^2} \right] q_1^4 + \left[ \frac{2-3n}{3a_1} \sum_{i=1}^n \sqrt{\frac{a_i}{a_1}} \right] q_1^3 + \left[ 2 \sum_{i=1}^n \sum_{j=i}^n \sqrt{\frac{a_i a_j}{a_1 a_1}} - \sum_{i=1}^n \frac{a_i}{a_1} \right] q_1^2 \end{aligned} \quad (57)$$

where we skip the mathematical manipulations in between. Finally, let  $q_1^0, q_2^0, \dots, q_n^0$  be the solution satisfying the first order conditions in (15) and (16). If for all  $i$ ,  $q_i^0 \leq a_i$ , then this solution is feasible and it is the unique optimal solution by Theorem 1. ■

**Proof. (Theorem 5):** Since  $A_i \sim \text{Exponential}(\mu_i)$ ,  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  can be written as in (20) and (21). Now, consider the optimality conditions given in Theorem 1. By using (20) and (21), (11) can be rewritten as

$$q_1 - \frac{1}{\mu_1} (1 - e^{-\mu_1 q_1}) = q_k - \frac{1}{\mu_k} (1 - e^{-\mu_k q_k}) \quad (58)$$

for any  $k$ . The solution of (58) gives (31) by using the Lambert  $W$  function. We substitute  $E[Y_{q_i}]$  and  $E[Y_{q_i}^2]$  in (10) to derive the first order optimality condition in (30). This leads to

$$\begin{aligned} \frac{2K_n D}{h} &= 2q_1 \sum_{i=1}^n \left( \frac{1}{\mu_i} \left( 1 + W \left( -e^{-1-\mu_i f(q_1)} \right) \right) \right) \\ &\quad + 2 \sum_{i=2}^n \sum_{j=i}^n \left( \frac{1}{\mu_i} \left( 1 + W \left( -e^{-1-\mu_i f(q_1)} \right) \right) \right) \left( \frac{1}{\mu_j} \left( 1 + W \left( -e^{-1-\mu_j f(q_1)} \right) \right) \right) \\ &\quad - \sum_{i=1}^n \left( \frac{2}{\mu_i^2} \left( 1 + \left( 2 + \mu_i f(q_1) + W \left( -e^{-1-\mu_i f(q_1)} \right) \right) W \left( -e^{-1-\mu_i f(q_1)} \right) \right) \right) \end{aligned}$$

which, after some tedious mathematical manipulations, can be shown to equal (30). ■

**Proof. (Theorem 6):** We check the first and second order optimality conditions to find the optimal order quantity that minimizes  $TC(q)$ . Differentiating (32) with respect to  $q$  gives

$$\frac{dTC(q)}{dq} = \frac{h\bar{F}(q)}{n^2 E[Y_q]^2} \left[ \left( nqE[Y_q] - \frac{n}{2}E[Y_q^2] + \frac{n(n-1)}{2}E[Y_q]^2 \right) - \frac{K_n D}{h} \right]. \quad (59)$$

Note that the denominator is always nonnegative and since  $\bar{F}(q) > 0$  for all  $q \geq 0$ , the second term of the numerator should be zero at the optimal solution which gives us the required equality in (33). Let us denote the left-hand side of (33) as

$$w(q) = 2nqE[Y_q] + n(n-1)E[Y_q]^2 - nE[Y_q^2]. \quad (60)$$

It is observed that  $w(0) = 0$  and  $w(q)$  is strictly increasing since  $w'(q) = 2nE[Y_q] + 2n(n-1)E[Y_q]\bar{F}(q) > 0$  for  $q > 0$ . Therefore, there is a unique and finite solution satisfying  $w(Q_n) = 2K_n D/h$ . The fact that this is the global minimum follows by noting that  $TC$  is unimodally decreasing on  $[0, Q_n]$  and increasing on  $[Q_n, +\infty]$ . This is a direct consequence of the fact that  $w(q)$  is strictly increasing so that the sign of the derivative of  $TC$  in (59) is the same as that of  $w(q) - (2K_n D/h)$ . ■

**Proof. (Corollary 7):** Assume  $n \geq n^*$  so that (34) is satisfied uniquely on  $[0, a]$  by the optimal order quantity. We can rewrite (36) as

$$\frac{n+1}{n}Q_{n+1} \leq Q_n \quad (61)$$

where  $w_n(Q_n) = w_{n+1}(Q_{n+1}) = 2KD/h$ . Since  $w_n(q)$  is increasing on  $[0, a]$ , it suffices to show that  $w_n(\frac{n+1}{n}Q_{n+1}) \leq w_n(Q_n) = 2KD/h$  to prove (36). We now evaluate  $w_n(q)$  for  $q = \frac{n+1}{n}Q_{n+1}$  and verify that, in fact,

$$\begin{aligned} w_n\left(\frac{n+1}{n}Q_{n+1}\right) &= \left[\frac{(n-1)}{4a^2}\right] \frac{(n+1)^4}{n^3} Q_{n+1}^4 + \left[\frac{(2-3n)}{3a}\right] \frac{(n+1)^3}{n^2} Q_{n+1}^3 \\ &\quad + (n+1)^2 Q_{n+1}^2 \leq \frac{2KD}{h} = w_{n+1}(Q_{n+1}). \end{aligned} \quad (62)$$

After writing the expression (34) for  $Q_{n+1}$  on the right-hand side of (62), the required inequality in (62) is true if and only if

$$\left[\frac{(n-1)(n+1)^4}{4a^2 n^3} - \frac{(n+1)n}{4a^2}\right] Q_{n+1}^4 + \left[\frac{(2-3n)(n+1)^3}{3an^2} + \frac{(3n+1)(n+1)}{3a}\right] Q_{n+1}^3 \leq 0. \quad (63)$$

In order to check if (63) holds, we define a function  $l(q)$  on  $[0, a]$  so that  $l(Q_{n+1})$  is given by the left-hand side of (63). We will show more generally that  $l(q) \leq 0$  for all  $0 \leq q \leq a$ . Note that  $l(0) = 0$  and

$$\frac{dl(q)}{dq} = \frac{(n+1)}{a} q^2 \left[ \frac{q}{a} \left( 2 - \frac{2}{n^2} - \frac{1}{n^3} \right) + \left( -3 + \frac{1}{n} + \frac{2}{n^2} \right) \right]. \quad (64)$$



In this expression, the first and second multiplicands are always nonnegative and the third multiplicand is nonpositive for  $0 \leq q \leq a$ . Hence,  $dl(q)/dq \leq 0$  on  $[0, a]$  and  $l(Q) \leq 0$  for  $0 \leq q \leq a$ .

To prove that the total order quantity decreases to the EOQ, take  $x_n = nQ_n$  and let  $x = \lim_{n \rightarrow +\infty} x_n$  be the limit of this decreasing sequence (for  $n \geq n^*$ ). Defining

$$\begin{aligned} b_n &= \left[ \frac{n(n-1)}{4a^2} \right] Q_n^4 + \left[ \frac{(2-3n)n}{3a} \right] Q_n^3 + n^2 Q_n^2 \\ &= \left[ \frac{x_n^2 - (x_n^2/n)x_n}{4a^2} \right] \left( \frac{x_n}{n} \right)^2 + \frac{2x_n}{3a} \left( \frac{x_n}{n} \right)^2 - \frac{x_n^2}{a} \left( \frac{x_n}{n} \right) + x_n^2 \end{aligned} \quad (65)$$

the result follows trivially from (34) by noting that  $\lim_{n \rightarrow +\infty} b_n = x^2$  for all values of  $a$  so that  $x = \text{EOQ} = \sqrt{2KD/h}$ . ■

**Proof. (Corollary 8):** We have already shown that (40) holds as an equality for  $n = 1$ . Suppose that  $n \geq 2$ . We can rewrite (40) as

$$\frac{n+1}{n} Q_{n+1} \leq Q_n \quad (66)$$

where  $Q_n$  and  $Q_{n+1}$  satisfy

$$\mu n Q_n - n(n-2)e^{-\mu Q_n} + 0.5n(n-1)e^{-2\mu Q_n} + 0.5n(n-3) = \frac{KD}{h} \mu^2 \quad (67)$$

and

$$\mu(n+1)Q_{n+1} - (n+1)(n-1)e^{-\mu Q_{n+1}} + 0.5(n+1)ne^{-2\mu Q_{n+1}} + 0.5(n+1)(n-2) = \frac{KD}{h} \mu^2 \quad (68)$$

from (39). Define

$$g(q) = \mu n q - n(n-2)e^{-\mu q} + 0.5n(n-1)e^{-2\mu q} + 0.5n(n-3) \quad (69)$$

which is clearly increasing on  $[0, +\infty)$  with  $g(0) = 0$ . So, to prove (66), it suffices to show that  $g(\frac{n+1}{n}Q_{n+1}) \leq g(Q_n)$ . By (67), we know that  $g(Q_n) = KD\mu^2/h$  is a constant. We now evaluate  $g(q)$  for  $q = \frac{n+1}{n}Q_{n+1}$  and verify that in fact

$$\begin{aligned} g\left(\frac{n+1}{n}Q_{n+1}\right) &= \mu(n+1)Q_{n+1} - n(n-2)e^{-\mu \frac{n+1}{n}Q_{n+1}} \\ &\quad + 0.5n(n-1)e^{-2\mu \frac{n+1}{n}Q_{n+1}} + 0.5n(n-3) \leq \frac{KD}{h} \mu^2 = g(Q_n). \end{aligned} \quad (70)$$

We insert (68) in (70) for  $KD\mu^2/h$  so that the required inequality in (70) is true if and only if

$$\begin{aligned} (n+1)(n-1)e^{-\mu Q_{n+1}} - 0.5n(n+1)e^{-2\mu Q_{n+1}} - n(n-2)e^{-\frac{n+1}{n}\mu Q_{n+1}} \\ + 0.5n(n-1)e^{-2\frac{n+1}{n}\mu Q_{n+1}} \leq n-1. \end{aligned} \quad (71)$$

In order to check if (71) holds, we define a function  $l(q)$  on  $[0, +\infty]$  so that  $l(Q_{n+1})$  is given by the left-hand side of (71). We will show more generally that  $l(q) \leq n - 1$  for all  $q \geq 0$ .

Note that  $l(0) = n - 1$  and  $\lim_{q \rightarrow +\infty} l(q) = 0 \leq n - 1$ . It is therefore sufficient to show that  $l(q)$  is a nonincreasing function for  $q \geq 0$ . Thus, we differentiate  $l(q)$  and check whether the derivative is nonpositive. Note that

$$\frac{dl(q)}{dq} = -\mu(n+1)e^{-\mu q} \left( (n-1) - ne^{-\mu q} - (n-2)e^{-\mu q/n} + (n-1)e^{-\mu q(n+2)/n} \right). \quad (72)$$

Apparently, (72) is nonpositive if

$$(n-1) - ne^{-\mu q} - (n-2)e^{-\mu q/n} + (n-1)e^{-\mu q(n+2)/n} \geq 0. \quad (73)$$

Following the same argument above, we see that the left-hand side in (73) is a function of  $q$  and takes values in the interval  $[0, n - 1]$ . Thus, it is sufficient to show that (73) is nondecreasing. So, we differentiate the left-hand side of (73) with respect to  $q$  and find that the derivative satisfies

$$\left(\frac{n-2}{n}\right)\mu e^{-\mu q/n} [1 - e^{-\mu q(n+1)/n}] + n\mu e^{-\mu q} [1 - e^{-2\mu q/n}] \geq 0 \quad (74)$$

since both terms in (74) are nonnegative for  $q \geq 0$ .

To prove that the total order quantity decreases to the EOQ, take  $x_n = nQ_n$  and let  $x = \lim_{n \rightarrow +\infty} x_n$  be the limit of this decreasing sequence. Defining

$$b_n = \frac{\mu x_n - n(n-2)e^{-(\frac{\mu}{n})x_n} + 0.5n(n-1)e^{-(\frac{2\mu}{n})x_n} + 0.5n(n-3)}{\mu^2} \quad (75)$$

the result follows trivially from (39) by noting that  $\lim_{n \rightarrow +\infty} b_n = x^2/2$  for all values of  $\mu$  so that  $x = \text{EOQ} = \sqrt{2KD/h}$ . The analysis starts by writing the exponential functions in (75) in series form using Taylor's expansion and evaluates the limit of each term to arrive at the quadratic function of  $x$ . ■

#### REFERENCES

- [1] Amupindi, R., and R. Akella, "Diversification Under Supply Uncertainty," *Management Science*, Vol. 39, No. 8, pp. 944–963, 1993.
- [2] Ciarallo, F. W., R. Akella, and T. E. Morton, "A Periodic Review, Production Planning Model with Uncertain Capacity and Uncertain Demand - Optimality of Extended Myopic Policies," *Management Science*, Vol. 40, No. 3, pp. 320–332, 1994.
- [3] Corless, R. M., G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert  $W$  Function", *Advances in Computational Mathematics*, Vol. 5, pp. 329–359, 1996.

- [4] Erdem, A., "Single-Period Inventory Models with Two Vendors and Random Yield", Ph. D. Dissertation, Department of Industrial Engineering, Boğaziçi University, 1999.
- [5] Erdem, A., S., and S. Özekici, "Inventory Models with Random Yield in a Random Environment", *International Journal of Production Economics*, Vol. 78, pp. 239-253, 2002.
- [6] Fadiloğlu, M. M., E. Berk, and M. C. Gürbüz, "An EOQ Model with Multiple Vendors and Binomial Yield", Technical Report, Department of Industrial Engineering, Bilkent University, 2005.
- [7] Gerchak, Y., and M. Parlar, "Yield Randomness, Cost Tradeoffs and Diversification in the EOQ Model," *Naval Research Logistics*, Vol. 37, pp. 341–354, 1990.
- [8] Güllü, R., "Base-stock Policies for Production/Inventory Problem with Uncertain Capacity Levels," *European Journal of Operational Research*, Vol. 105, pp. 43–51, 1998.
- [9] Gürler, Ü., and M. Parlar, "An Inventory Problem with Two Randomly Available Suppliers," *Operations Research*, Vol. 45, pp. 904–918, 1997.
- [10] Henig, M., and Y. Gerchak, "The Structure of Periodic Review Policies in the Presence of Random Yield," *Operations Research*, Vol. 38, No. 4, pp. 634–643, 1990.
- [11] Karlin, S., "One Stage Models with Uncertainty," in K. J. Arrow, S. Karlin and H. Scarf (Eds.) *Studies in the Mathematical Theory of Inventory and Production*, pp. 109–134, Stanford: Stanford University Press, 1958.
- [12] Lee, H. L., and C. A. Yano, "Production Control in Multistage Systems with Variable Yield Losses," *Operations Research*, Vol. 36, No. 2, pp. 269–278, 1988.
- [13] Özekici, S., and M. Parlar, "Inventory Models with Unreliable Suppliers in a Random Environment," *Annals of Operations Research*, Vol. 91, pp. 123–136, 1999.
- [14] Parlar, M., and D. Berkin, "Future Supply Uncertainty in EOQ Models," *Naval Research Logistics*, Vol. 38, pp. 107–121, 1991.
- [15] Parlar, M., and D. Wang, "Diversification Under Yield Randomness in Inventory Models," *European Journal of Operational Research*, Vol. 66, pp. 52–64, 1993.
- [16] Shih, W., "Optimal Inventory Policies When Stockouts Result From Defective Products," *International Journal of Production Research*, Vol. 18, pp. 677–686, 1980.

- [17] Silver, E. A., “Establishing the Order Quantity When the Amount Received is Uncertain,” *INFOR*, Vol. 14, pp. 32–39, 1976.
- [18] Wang, Y., and Y., Gerchak, “Periodic Review Production Models with Variable Capacity, Random Yield and Uncertain Demand”, *Management Science*, Vol. 42, No. 1, pp.130-137, 1996a.
- [19] Wang, Y., and Y., Gerchak, “Continuous Review Inventory Control When Capacity is Variable”, *International Journal of Production Economics*, Vol. 45, pp. 381-388, 1996b.
- [20] Yano, C. A., and H. L. Lee, “Lot Sizing with Random Yields: A Review,” *Operations Research*, Vol. 43, No. 2, pp. 311–334, 1995.